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On gradient Schouten solitons conformal to a pseudo-Euclidean space

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Abstract. In this paper we consider ρ -Einstein solitons that are conformal to a pseudo-Euclidean space and invariant under the action of the pseudo-orthogonal group. We provide all the solutions for the gradient Schouten soliton case. Moreover, we proved that if a gradient Schouten soliton is both complete, conformal to a Euclidean metric, and rotationally symmetric, then it is isometric to $\mathbb{R} \times \mathbb{S}^{n-1}$.

1. Introduction and main statements

In 1982 Hamilton [1] introduced the Ricci flow equation

$$\frac{\partial}{\partial t} g(t) = -2Ric(g(t))$$

where $Ric(g(t))$ is the Ricci curvature of the metric $g(t)$. A important aspect in the treatment of the Ricci flow is the study of Ricci solitons, which generate self-similar solutions to the flow and often arise as singularity models.

Given a semi-Riemannian manifold (M^n, g) , $n \geq 3$, we say that (M, g) is a gradient Ricci soliton if there exists a differentiable function $f : M \rightarrow \mathbb{R}$ (called the potential function) such that

$$Ric_g + Hess_g(f) = \lambda g, \quad \lambda \in \mathbb{R},$$

where Ric_g is the Ricci tensor, $Hess_g(f)$ is the Hessian of f with respect to the metric g , and λ is a real number.

We say that a gradient Ricci soliton is *shrinking, steady, or expanding* if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. Bryant [2] proved that there exists a complete, steady, gradient Ricci soliton that is spherically symmetric for any $n \geq 3$, which is known as Bryant's soliton. In the bi-dimensional case an analogous nontrivial rotationally symmetric solution was obtained explicitly, and is known as the Hamilton cigar. Recently Cao and Chen [3] showed that any complete, steady, gradient Ricci soliton, locally conformally flat, up to homothety, is either flat or isometric to the Bryant's

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soliton. The results obtained in [3] were extended to bach - flat gradient steady Ricci solitons (see [4]). Complete, conformally flat shrinking gradient solitons have been characterized as being quotients of $\mathbb{R}^n$, $\mathbb{S}^n$ or $R \times \mathbb{S}^{n-1}$ (see [5]).

In the case of steady gradient Ricci solitons, [6] provide all such solutions when the metric is conformal to an n -dimensional pseudo-Euclidean space and invariant under the action of an $(n - 1)$ -dimensional translation group.

Motivated by the notion of Ricci solitons on a semi-Riemannian manifold (M^n, g) , $n \geq 3$, it is natural to consider geometric flows of the following type:

$$\frac{\partial}{\partial t} g(t) = -2(Ric - \rho Rg) \quad (1)$$

for $\rho \in \mathbb{R}$, $\rho \neq 0$, as in [7]. We call these the Ricci–Bourguignon flows. We notice that short time existence for the geometric flows described in (1) is provided in ([8]). Associated to the flows, we have the following notion of gradient ρ -Einstein solitons, which generate self-similar solutions:

Definition 1.1. Let (M^n, g) , $n \geq 3$, be a Riemannian manifold and let $\rho \in \mathbb{R}$, $\rho \neq 0$. We say that (M^n, g) is a gradient ρ -Einstein soliton if there exists a smooth function $f : M \rightarrow \mathbb{R}$, such that the metric g satisfies the equations

$$Ric_g + Hess_g f = \rho Rg + \lambda g \quad (2)$$

for some constant $\lambda \in \mathbb{R}$, where R is the scalar curvature of the metric g .

A ρ -Einstein soliton is said to be shrinking, steady, or expanding if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. Furthermore, a ρ -Einstein solitons is said to be a gradient Einstein soliton, gradient traceless Ricci soliton, and gradient Schouten soliton if $\rho = \frac{1}{2}$, $\rho = \frac{1}{n}$, and $\rho = \frac{1}{2(n-1)}$, respectively.

In [7] the authors studied the ρ -Einstein solitons and obtained important rigidity results, proving that every compact gradient Einstein, Schouten, or traceless Ricci soliton is trivial.

In this paper we study Eq. (2) in semi-Riemannian manifolds. We consider gradient ρ -Einstein solitons conformal to a pseudo-Euclidean space, which are invariant under the action of the pseudo-orthogonal group. More precisely, let $(\mathbb{R}^n, g)$ be the standard pseudo-Euclidean space with metric g and coordinates $(x_1, \dots, x_n)$, with $g_{ij} = \delta_{ij}\varepsilon_i$, $1 \leq i, j \leq n$, where δ_{ij} is the Kronecker delta, and $\varepsilon_i = \pm 1$, with at least one ε_i equal to one. Let $r = \sum_{i=1}^n \varepsilon_i x_i^2$ be a basic invariant for an $(n - 1)$ -dimensional pseudo-orthogonal group. The main objective of this paper is to found the necessary and sufficient conditions for the existence of differentiable functions $\varphi(r)$ and $f(r)$ such that the metric $\bar{g} = \frac{g}{\varphi^2}$ satisfies the equation

$$Ric_{\bar{g}} + Hess_{\bar{g}} f = \rho \bar{R}\bar{g} + \lambda \bar{g} \quad (3)$$

with $\rho = 1/2(n - 1)$.

We initially find a system of differential equations, that functions f and φ must satisfy, so that the metric $\bar{g} = g/\varphi^2$ satisfies (3) (see Theorem 1.1).

Note that if the solutions are invariant under the action of the pseudo-orthogonal group, the system of partial differential equations given in Theorem 1.1 can be transformed into a system of ordinary differential equations (see Theorem 1.1).

Considering $\rho = \frac{1}{2(n-1)}$, in Lemma 1.1 we found conditions necessary for the functions to be solutions of the system of ODE in Corollary 1.1. Finally, we obtain the necessary and sufficient conditions on $f(r)$ and $\varphi(r)$ for the existence of one gradient Schouten soliton. In this case, all solutions are given explicitly. Moreover, in the Riemannian case, we prove that if $(M^n, g), n \geq 3$ is complete, conformal to a Euclidean metric, rotationally symmetric, and satisfying

$$\text{Ric}_{\bar{g}} + \text{Hess}_{\bar{g}} f = \frac{\bar{R}}{2(n-1)} \bar{g} + \lambda \bar{g} \quad (4)$$

for same $\lambda \in \mathbb{R}$, then (M^n, g) is shrinking and isometric to $\mathbb{R} \times \mathbb{S}^{n-1}$. In [7], the authors showed that any complete metric must be Ricci flat if it satisfies (4) for $\lambda = 0$.

Remark 1.1. Note that if $\lambda = 0$ and $\rho = 0$, Eq. (3) represents a steady gradient Ricci soliton that admits solutions.

In what follows, we state our main results. We denote the second order derivative of φ and f by $\varphi_{,x_i x_j}$ and $f_{,x_i x_j}$, respectively, with respect to $x_i x_j$.

Theorem 1.1. Let $(\mathbb{R}^n, g), n \geq 3$, be a pseudo-Euclidean space with coordinates $x = (x_1, \dots, x_n)$ and metric components $g_{ij} = \delta_{ij} \varepsilon_i$, $1 \leq i, j \leq n$, where $\varepsilon_i = \pm 1$, with at least one ε_i equal to one. Consider a smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. There exists a metric $\bar{g} = \frac{1}{\varphi^2} g$ such that $(\mathbb{R}^n, \bar{g})$ is a gradient ρ -Einstein soliton with f as a potential function if, and only if, the functions φ , and f satisfy

$$(n-2)\varphi_{,x_i x_j} + \varphi f_{,x_i x_j} + \varphi_{,x_i} f_{,x_j} + \varphi_{,x_j} f_{,x_i} = 0, \quad i \neq j, \quad (5)$$

$$\begin{aligned} & \varphi \left[(n-2)\varphi_{,x_i x_i} + \varphi f_{,x_i x_i} + 2\varphi_{,x_i} f_{,x_i} \right] \\ & + \varepsilon_i \sum_{k=1}^n \varepsilon_k \left[(n-1) \left(\rho n \varphi_{,x_k}^2 - 2\rho \varphi \varphi_{,x_k x_k} - \varphi_{,x_k}^2 \right) \right. \\ & \left. - \varphi \varphi_{,x_k} f_{,x_k} + \varphi \varphi_{,x_k x_k} \right] = \lambda \varepsilon_i, \quad i = j. \end{aligned} \quad (6)$$

Our objective is to determine solutions of the system (5), (6) of the form $\varphi(r)$ and $f(r)$, where $r = \sum_{i=1}^n \varepsilon_i x_i^2$. The following theorem reduces the system of partial differential equations (5) and (6) into an equivalent system of ordinary differential equations.

Corollary 1.1. Let $(\mathbb{R}^n, g), n \geq 3$, be a pseudo-Euclidean space with coordinates $x = (x_1, \dots, x_n)$ and metric components $g_{ij} = \delta_{ij} \varepsilon_i$, $1 \leq i, j \leq n$, where $\varepsilon_i = \pm 1$, with at least one ε_i equal to 1. Consider smooth functions $\varphi(r)$ and $f(r)$ with $r = \sum_{k=1}^n \varepsilon_k x_k^2$. Then there exists a metric $\bar{g} = \frac{1}{\varphi^2} g$ such that $(\mathbb{R}^n, \bar{g})$ is a gradient ρ -Einstein soliton with f as a potential function if, and only if, the functions φ and f satisfy

$$(n-2)\varphi'' + \varphi f'' + 2\varphi' f' = 0, \quad (7)$$

and

$$\begin{aligned} 2\varphi [(n-2)\varphi' + \varphi f'] + 2n[1 - 2(n-1)\rho]\varphi\varphi' \\ + 4r \left\{ (n-1) \left[(\rho n - 1)(\varphi')^2 - 2\rho\varphi\varphi'' \right] \right. \\ \left. - \varphi\varphi' f' + \varphi\varphi'' \right\} = \lambda. \end{aligned} \quad (8)$$

When $\rho = \frac{1}{2(n-1)}$, we obtain the following result:

Corollary 1.2. *Let $(\mathbb{R}^n, g)$, $n \geq 3$, be a pseudo-Euclidean space with coordinates $x = (x_1, \dots, x_n)$ and metric components $g_{ij} = \delta_{ij}\varepsilon_i$, $1 \leq i, j \leq n$, where $\varepsilon = \pm 1$, with at least one ε_i equal to one. Consider smooth functions $\varphi(r)$ and $f(r)$ with $r = \sum_{k=1}^n \varepsilon_k x_k^2$. Then there exists a metric $\bar{g} = \frac{1}{\varphi^2}g$ such that $(\mathbb{R}^n, \bar{g})$ is a gradient Schouten soliton with f as a potential function if, and only if, the functions φ and f satisfy*

$$(n-2)\varphi'' + \varphi f'' + 2\varphi' f' = 0, \quad (9)$$

and

$$\varphi [(n-2)\varphi' + \varphi f'] + r \left[(2-n)(\varphi')^2 - 2\varphi\varphi' f' \right] = \frac{\lambda}{2}. \quad (10)$$

The following lemma gives the necessary conditions to have solutions for the system (9) and (10).

Lemma 1.1. *Consider smooth functions $\varphi(r)$ and $f(r)$ such that $r = \sum_{i=1}^n \varepsilon_i x_i^2$. If $\varphi(r)$ and $f(r)$ satisfy (9) and (10), then $\varphi(r)$ satisfies the following ordinary differential equation:*

$$r\varphi\varphi'' = r(\varphi')^2 - \varphi\varphi'.$$

As a consequence of Lemma 1.1, we found all the conformal metrics, invariant by the action of the pseudo-orthogonal group, that are gradient Schouten solitons.

Theorem 1.2. *Let $(\mathbb{R}^n, g)$, $n \geq 3$, be a pseudo-Euclidean space with coordinates $x = (x_1, \dots, x_n)$ and metric component $g_{ij} = \delta_{ij}\varepsilon_i$, $1 \leq i, j \leq n$, where $\varepsilon = \pm 1$, with at least one ε_i equal to one. Consider smooth functions $\varphi(r)$ and $f(r)$ such that $r = \sum_{k=1}^n \varepsilon_k x_k^2$. Then there exists a metric $\bar{g} = \frac{1}{\varphi^2}g$ such that $(\mathbb{R}^n, \bar{g})$ is a gradient Schouten soliton with f as a potential function if, and only if, the functions φ and f are given by*

$$\begin{cases} \varphi(r) = kr^s, \\ f(r) = \frac{(2-n)(1-s)s}{(1-2s)} \ln r + \frac{\lambda r^{1-2s}}{2k^2(1-2s)^2} + k_1, & \text{for } s \neq \frac{1}{2}, \\ f(r) = \frac{(n-2)}{8} (\ln r)^2 + c \ln r + c_1, & \text{for } s = \frac{1}{2}, \end{cases}$$

where $s \in \mathbb{R}_+^*$ and $c, c_1 \in \mathbb{R}$.

Corollary 1.3. *Let $(M^n, \bar{g})$, $n \geq 3$, a complete gradient Schouten soliton, conformal to a Euclidean metric and rotationally symmetric, where $\bar{g}_{ij} = \delta_{ij}/\varphi^2$. Then $(M^n, \bar{g})$ is shrinking and isometric to $\mathbb{R} \times \mathbb{S}^{n-1}$.*

In the case $n = 2$, Eq. (4) reduces to the gradient Yamabe solitons equation, i.e.,

$$\text{Hess}_{\bar{g}} f = \left(-\frac{\bar{R}_{\bar{g}}}{2} + \lambda \right) \bar{g} \quad (11)$$

and in this case the metric $\bar{g}$ is a gradient Yamabe soliton (see [9] and [10]). In Theorem 1.3 we find all solutions of (11) that are conformal to the pseudo-Euclidean metric and invariant under the action of the pseudo-orthogonal group.

Theorem 1.3. *Let $(\mathbb{R}^2, g)$, be a pseudo-Euclidean space with coordinates $x = (x_1, x_2)$ and $g_{ij} = \delta_{ij}\varepsilon_i$. Consider smooth functions $\varphi(r)$ and $f(r)$ with $r = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2$. There exists a metric $\bar{g} = \frac{1}{\varphi^2} g$ such that $(\mathbb{R}^2, \bar{g})$ is a gradient Yamabe soliton with f as a potential function if, and only if, the functions φ and f satisfy*

$$\begin{cases} \varphi(r) = k_1 r^{\left(\frac{2k-\lambda}{4k}\right)}, \\ f(r) = \frac{2k^2}{k_1^2 \lambda} r^{\left(\frac{\lambda}{2k}\right)} + c, \text{ for } \lambda \neq 0 \\ f(r) = k_2 \ln r + k_3 \text{ for } \lambda = 0, \end{cases}$$

where $k, k_1, k_2 \in \mathbb{R}_+^*$ and $k_3, c \in \mathbb{R}$.

In the Riemannian case and with $\lambda = 0$, the metrics obtained in Theorem 1.3 are complete and isometric to $\mathbb{R} \times \mathbb{S}^1$.

The last result appears in [9].

2. Proofs of the main results

Proof of Theorem 1.1. It is well known (see, e.g., [6]) that if $\bar{g} = \frac{g}{\varphi^2}$, then

$$\text{Ric}_{\bar{g}} = \frac{1}{\varphi^2} \{ (n-2)\varphi \text{Hess}_g(\varphi) + [\varphi \Delta_g \varphi - (n-1)|\nabla_g \varphi|^2] g \}$$

and

$$\bar{R} = (n-1) \left(2\varphi \Delta_g \varphi - n|\nabla_g \varphi|^2 \right).$$

Hence, the equation

$$\text{Ric}_{\bar{g}} + \text{Hess}_{\bar{g}}(f) = \rho \bar{R} \bar{g} + \lambda \bar{g},$$

is equivalent to

$$\begin{aligned} & \frac{1}{\varphi^2} \{ (n-2)\varphi \text{Hess}_g(\varphi)_{ij} + [\varphi \Delta_g \varphi - (n-1)|\nabla_g \varphi|^2] \delta_{ij} \varepsilon_i \} + \text{Hess}_{\bar{g}}(f)_{ij} \\ &= \left[\rho(n-1)(2\varphi \Delta_g \varphi - n|\nabla_g \varphi|^2) + \lambda \right] \frac{1}{\varphi^2} \delta_{ij} \varepsilon_i. \end{aligned} \quad (12)$$

Recall that,

$$Hess_{\bar{g}}(f)_{ij} = f_{,x_i x_j} - \sum_{k=1}^n \bar{\Gamma}_{ij}^k f_{,x_k}$$

where $\bar{\Gamma}_{ij}^k$ are the Christoffel symbols of the metric $\bar{g}$. For a distinct i, j, k , we have

$$\bar{\Gamma}_{ij}^k = 0, \quad \bar{\Gamma}_{ij}^i = -\frac{\varphi_{,x_j}}{\varphi}, \quad \bar{\Gamma}_{ii}^k = \varepsilon_i \varepsilon_k \frac{\varphi_{,x_k}}{\varphi}, \quad \bar{\Gamma}_{ii}^i = -\frac{\varphi_{,x_i}}{\varphi},$$

therefore,

$$Hess_{\bar{g}}(f)_{ij} = f_{,x_i x_j} + \frac{\varphi_{,x_j} f_{,x_i}}{\varphi} + \frac{\varphi_{,x_i} f_{,x_j}}{\varphi}, \quad i \neq j. \quad (13)$$

Similarly, by considering $i = j$, we have

$$Hess_{\bar{g}}(f)_{ii} = f_{,,x_i x_i} + \frac{2\varphi_{,x_i} f_{,x_i}}{\varphi} - \varepsilon_i \sum_{k=1}^n \varepsilon_k \frac{\varphi_{,x_k} f_{,x_k}}{\varphi}. \quad (14)$$

However, we note that

$$|\nabla_g \varphi|^2 = \sum_{k=1}^n \varepsilon_k \left(\frac{\partial \varphi}{\partial x_k} \right)^2, \quad \Delta_g \varphi = \sum_{k=1}^n \varepsilon_k \varphi_{,x_k x_k}, \quad Hess_g(\varphi)_{ij} = \varphi_{,x_i x_j}. \quad (15)$$

If $i \neq j$ in (12), we obtain

$$(n-2) \frac{Hess_g(\varphi)_{ij}}{\varphi} + Hess_{\bar{g}}(f)_{ij} = 0. \quad (16)$$

Substituting the expressions found in (13), and (15) into (16), we obtain

$$(n-2)\varphi_{,x_i x_j} + \varphi f_{,x_i x_j} + \varphi_{,x_i} f_{,x_j} + \varphi_{,x_j} f_{,x_i} = 0, \quad i \neq j.$$

Similarly, if $i = j$ in (12), we have

$$\begin{aligned} (n-2)\varphi Hess_g(\varphi)_{ii} + \varphi \Delta_g \varphi \varepsilon_i - (n-1)|\nabla_g \varphi|^2 \varepsilon_i + \varphi^2 Hess_{\bar{g}}(f)_{ii} \\ = 2(n-1)\rho \Delta_g \varphi \varepsilon_i - n(n-1)\rho |\nabla_g \varphi|^2 \varepsilon_i + \lambda \varepsilon_i. \end{aligned} \quad (17)$$

Substituting the expressions found in (14), and (15) into (17), we obtain

$$\begin{aligned} \varphi [(n-2)\varphi_{,x_i x_i} + \varphi f_{,x_i x_i} + 2\varphi_{,x_i} f_{,x_i}] \\ + \varepsilon_i \sum_{k=1}^n \varepsilon_k \left[(n-1) \left(\rho n \varphi_{,x_k}^2 - 2\rho \varphi \varphi_{,x_k x_k} - \varphi_{,x_k}^2 \right) \right. \\ \left. - \varphi \varphi_{,x_k} f_{,x_k} + \varphi \varphi_{,x_k x_k} \right] = \lambda \varepsilon_i. \end{aligned}$$

This concludes the proof of Theorem 1.1. $\square$

Proof of Corollary 1.1. Let $\bar{g} = \varphi^{-2}g$ be a conformal metric of g . We are assuming that $\varphi(r)$ and $f(r)$ are functions of r , where $r = \sum_{k=1}^n \varepsilon_k x_k^2$. Hence, we have

$$\varphi_{,x_i} = 2\varepsilon_i x_i \varphi', \quad \varphi_{,x_i x_i} = 4x_i^2 \varphi'' + 2\varepsilon_i \varphi', \quad \varphi_{,x_i x_j} = 4\varepsilon_i \varepsilon_j x_i x_j \varphi''$$

and

$$f_{,x_i} = 2\varepsilon_i x_i f', \quad f_{,x_i x_i} = 4x_i^2 f'' + 2\varepsilon_i f', \quad f_{,x_i x_j} = 4\varepsilon_i \varepsilon_j x_i x_j f''.$$

Substituting these expressions into (5), we obtain

$$4\varepsilon_i \varepsilon_j (n-2)x_i x_j \varphi'' + 4\varepsilon_i \varepsilon_j x_i x_j \varphi f'' + (2\varepsilon_i x_i \varphi').(2\varepsilon_j x_j f') \\ + (2\varepsilon_j x_j \varphi').(2\varepsilon_i x_i f') = 0,$$

which is equivalent to

$$4\varepsilon_i \varepsilon_j [(n-2)\varphi'' + \varphi f'' + 2\varphi' f'] x_i x_j = 0.$$

Since there exist $i \neq j$, such that $x_i x_j \neq 0$, we have

$$(n-2)\varphi'' + \varphi f'' + 2\varphi' f' = 0.$$

Similarly, considering the Eq. (6), we obtain

$$4\varphi [(n-2)\varphi'' + \varphi f'' + 2\varphi' f'] x_i^2 + 2\varphi [(n-2)\varphi' + \varphi f'] \varepsilon_i \\ + 2\varepsilon_i n[1 - 2(n-1)\rho]\varphi\varphi' \\ + 4\varepsilon_i \sum_{k=1}^n \varepsilon_k x_k^2 \left\{ (n-1) [(\rho n - 1)(\varphi')^2 - 2\rho\varphi\varphi''] - \varphi\varphi' f' + \varphi\varphi'' \right\} = \lambda \varepsilon_i.$$

Note that $(n-2)\varphi'' + \varphi f'' + 2\varphi' f' = 0$ and $r = \sum_{k=1}^n \varepsilon_k x_k^2$. Therefore, we obtain

$$2\varphi [(n-2)\varphi' + \varphi f'] + 2n[1 - 2(n-1)\rho]\varphi\varphi' \\ + 4r \left\{ (n-1) [(\rho n - 1)(\varphi')^2 - 2\rho\varphi\varphi''] - \varphi\varphi' f' + \varphi\varphi'' \right\} = \lambda.$$

This concludes the proof of Corollary 1.1. $\square$

Proof of Corollary 1.2. Simply substitute $\rho = \frac{1}{2(n-1)}$ into Eq. (8), and the result follows. $\square$

Proof of Lemma 1.1. Differentiating both sides of the Eq. (10), we obtain

$$\varphi' [(n-2)\varphi' + \varphi f'] + \varphi [(n-2)\varphi'' + \varphi' f' + \varphi f''] + [(2-n)(\varphi')^2 - 2\varphi\varphi' f'] \\ r \left\{ 2(2-n)\varphi' \varphi'' - 2[(\varphi')^2 f' + \varphi\varphi'' f' + \varphi\varphi' f''] \right\} = 0.$$

This is equivalent to

$$(n-2)(\varphi')^2 + \varphi\varphi'f' + (n-2)\varphi\varphi'' + \varphi\varphi'f' + \varphi^2f'' + (2-n)(\varphi')^2 - 2\varphi\varphi'f' + 2(2-n)r\varphi\varphi'' - 2r(\varphi')^2f' - 2r\varphi\varphi''f' - 2r\varphi\varphi'f'' = 0. \quad (18)$$

By the first equation in Theorem 1.1, we obtain

$$\varphi f'' = -[(n-2)\varphi'' + 2\varphi'f']. \quad (19)$$

Substituting (19) into (18), we obtain

$$(n-2)\varphi\varphi'' - \varphi[(n-2)\varphi'' + 2\varphi'f'] + 2(2-n)r\varphi\varphi'' - 2r(\varphi')^2f' - 2r\varphi\varphi''f' + 2r\varphi'[(n-2)\varphi'' + 2\varphi'f'] = 0,$$

which is equivalent to

$$2\varphi\varphi'f' - 2r(\varphi')^2f' + 2r\varphi\varphi''f' = 0.$$

How $f' \neq 0$, we have that

$$r\varphi\varphi'' = r(\varphi')^2 - \varphi\varphi'.$$

This concludes the proof of Lemma 1.1. $\square$

Proof of Theorem 1.2. We know by Lemma 1.1, that the conformal factor necessarily satisfies the equation

$$r\varphi\varphi'' = r(\varphi')^2 - \varphi\varphi'.$$

Dividing the above equation by $r\varphi^2$, we can verify that

$$\left(\frac{\varphi'}{\varphi}\right)' = -\frac{1}{r}\frac{\varphi'}{\varphi}.$$

Making $y = \frac{\varphi'}{\varphi}$ we obtain

$$y = sr^{-1}$$

where $s \in \mathbb{R}_+^*$. Therefore

$$\varphi = kr^s, \quad (20)$$

where $k \in \mathbb{R}_+^*$.

Since φ and f satisfy Eq. (10), by substituting (20) into (10), we obtain

$$kr^s \left[(n-2)ksr^{s-1} + kr^sf' \right] + r \left[(2-n)k^2s^2r^{2s-2} - 2kr^sksr^{s-1}f' \right] = \frac{\lambda}{2}$$

which is equivalent to,

$$\left[k^2r^{2s} - 2k^2sr^{2s} \right] f' + (n-2)k^2sr^{2s-1} + (2-n)k^2s^2r^{2s-1} = \frac{\lambda}{2}.$$

If $s \neq \frac{1}{2}$, then

$$f'(r) = \frac{(2-n)(1-s)s}{(1-2s)r} + \frac{\lambda}{2k^2(1-2s)r^{2s}}.$$

Consequently,

$$f(r) = \frac{(2-n)(1-s)s}{(1-2s)} \ln r + \frac{\lambda r^{1-2s}}{2k^2(1-2s)^2} + k_1. \quad (21)$$

Reciprocally, the function $f(r)$ and $\varphi(r)$ given in (20) and (21) satisfy (9) and (10). Note that

$$\varphi = kr^s, \quad \varphi' = ksr^{s-1}, \quad \varphi'' = ks(s-1)r^{s-2}$$

and

$$\begin{aligned} f'(r) &= \frac{(2-n)(1-s)s}{(1-2s)r} + \frac{\lambda}{2k^2(1-2s)r^{2s}}, \\ f''(r) &= -\left(\frac{(2-n)(1-s)s}{(1-2s)r^2} + \frac{\lambda s}{k^2(1-2s)r^{2s+1}} \right). \end{aligned}$$

In this case

$$\begin{aligned} (n-2)\varphi'' + f''\varphi + 2\varphi'f' &= (n-2)ks(s-1)r^{s-2} - \left(\frac{(2-n)(1-s)s}{(1-2s)r^2} \right. \\ &\quad \left. + \frac{\lambda s}{k^2(1-2s)r^{2s+1}} \right) kr^s \\ &\quad + 2ksr^{s-1} \left(\frac{(2-n)(1-s)s}{(1-2s)r} + \frac{\lambda}{2k^2(1-2s)r^{2s}} \right) \end{aligned}$$

equivalently,

$$\begin{aligned} (n-2)\varphi'' + f''\varphi + 2\varphi'f' &= \frac{(n-2)(s-1)ks}{r^{2-s}} - \frac{2(2-n)(s-1)s^2k}{(1-2s)r^{2-s}} \\ &\quad + \frac{(2-n)(s-1)sk}{(1-2s)r^{2-s}}. \end{aligned}$$

Therefore,

$$(n-2)\varphi'' + f''\varphi + 2\varphi'f' = 0.$$

Meanwhile, substituting φ and f into (10), we obtain

$$\begin{aligned} \varphi \left[(n-2)\varphi' + \varphi f' \right] + r \left[(2-n)(\varphi')^2 - 2\varphi\varphi'f' \right] &= (1-2s) \frac{\lambda k^2 r^{2s}}{2k^2(1-2s)r^{2s}} \\ &\quad + (1-2s) \left[\frac{(2-n)(1-s)sk^2 r^{2s}}{(1-2s)r} \right] + (1-s)(n-2)k^2 sr^{2s-1}. \end{aligned}$$

Therefore,

$$\varphi[(n-2)\varphi' + \varphi f'] + r \left[(2-n)(\varphi')^2 - 2\varphi\varphi'f' \right] = \frac{\lambda}{2} + (2-n)(1-s)k^2r^{2s-1} \\ + (1-s)(n-2)k^2sr^{2s-1}$$

equivalently,

$$\varphi[(n-2)\varphi' + \varphi f'] + r \left[(2-n)(\varphi')^2 - 2\varphi\varphi'f' \right] = \frac{\lambda}{2}.$$

If $s = \frac{1}{2}$, we have $\lambda = \frac{(n-2)k^2}{2}$, and (10) is satisfied for all f . Substituting (20) in (9), we obtain $f(r) = \frac{(n-2)}{8}(\ln r)^2 + c \ln r + c_1$, where $c, c_1 \in \mathbb{R}$.

This concludes the proof of Theorem 1.2. $\square$

Proof of Corollary 1.3. We split the proof into three cases $s > \frac{1}{2}$, $s = \frac{1}{2}$ and $s < \frac{1}{2}$.

Case 1 If $s > \frac{1}{2}$, consider a divergent curve $\alpha(t) = (t, 0, \dots, 0)$. Note that $\alpha'(t) = (1, 0, \dots, 0)$, $\varphi(\alpha(t)) = t^{2s}$ and $\bar{g}(\alpha'(t), \alpha'(t)) = \frac{1}{t^{4s}}$. Consequently,

$$l(\alpha) = \lim_{t \rightarrow \infty} \int_2^t |\alpha'(t)| dt = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{t^{2s}} dt.$$

Therefore,

$$l(\alpha) = \lim_{t \rightarrow \infty} \left(\frac{t^{1-2s}}{1-2s} \right) + k = k,$$

where k is a constant (finite). Therefore $(\mathbb{R}^n \setminus \{0\}, \bar{g})$ is not complete.

Case 2 If $s < \frac{1}{2}$, we obtain

$$l(\alpha) = \lim_{t \rightarrow 0} \int_t^2 |\alpha'(t)| dt = \lim_{t \rightarrow 0} \int_2^t \frac{1}{t^{2s}} dt.$$

Therefore,

$$l(\alpha) = \lim_{t \rightarrow 0} \left(\frac{t^{1-2s}}{1-2s} \right) + k = k,$$

where k is a finite constant. Therefore $(\mathbb{R}^n \setminus \{0\}, \bar{g})$ is not complete.

Case 3 If $s = \frac{1}{2}$ the authors in [11] showed that $(\mathbb{R}^3 \setminus \{0\}, \bar{g} = \frac{1}{r}g_0)$ is isometric to $\mathbb{R} \times \mathbb{S}^2$, under the identification $\Psi : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R} \times \mathbb{S}^2$ given by $\Psi(x) = (\log \|x\|, x/\|x\|)$. Similarly we obtain that $(\mathbb{R}^n \setminus \{0\}, \bar{g} = \frac{1}{r}g_0)$ is isometric to $\mathbb{R} \times \mathbb{S}^{n-1}$, which is complete.

By expression (10), we obtain

$$\left[\varphi^2 - 2r\varphi\varphi' \right] f' + (n-2)\varphi\varphi' + (2-n)r(\varphi')^2 = \frac{\lambda}{2}.$$

Note that $\varphi(r) = kr^{\frac{1}{2}}$, therefore

$$\lambda = \frac{(n-2)k^2}{2} > 0.$$

Since $n \geq 3$, the proof is complete. $\square$

Proof of Theorem 1.3. From expression (9) in Corollary 1.2, we have

$$\frac{f''}{f'} = \frac{-2\varphi'}{\varphi}.$$

Consequently,

$$f' = \frac{k}{\varphi^2} \quad (22)$$

where $k \in \mathbb{R}_+^*$ is a constant of the integration.

Meanwhile, from expression (10) in Corollary 1.2, we obtain

$$2\varphi^2 f' - 4r\varphi\varphi' f' = \lambda. \quad (23)$$

Combining (22) and (23), we obtain

$$\begin{cases} \varphi(r) = k_1 r^{\left(\frac{2k-\lambda}{4k}\right)}, \\ f(r) = \frac{2k^2}{k_1^2 \lambda} r^{\frac{\lambda}{2k}} + c \end{cases}$$

where $k, k_1 \in \mathbb{R}_+^*$ and $c \in \mathbb{R}$.

Therefore the proof is complete for the case $\lambda \neq 0$. For the case $\lambda = 0$ the proof is analogous. $\square$

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