



Exact Solutions of Einstein Field Equation in Locally Conformally Flat Manifolds

Levi Rosa Adriano, Ilton Ferreira de Menezes,
Mauricio Donizetti Pieterzack, and Romildo da Silva Pina 

Abstract. We consider the pseudo-Euclidean space $(\mathbb{R}^n, g)$, $n \geq 3$, with coordinates $x = (x_1, \dots, x_n)$ and metric components $g_{ij} = \delta_{ij}\epsilon_i$, $1 \leq i, j \leq n$, where $\epsilon_i = \pm 1$, with at least one $\epsilon_i = 1$ and one diagonal $(0,2)$ -tensors of the form $T = \sum_i \epsilon_i h_i(x) dx_i^2$. We obtain necessary and sufficient conditions for the existence of a metric $\bar{g}$, conformal to g , such that $\text{Ric}_{\bar{g}} - \frac{\bar{K}}{2}\bar{g} = T$, where $\text{Ric}_{\bar{g}}$ and $\bar{K}$ are the Ricci tensor and scalar curvature of the metric $\bar{g}$, respectively. Using the results obtained, we construct an example of a static perfect fluid spacetime. Similar problems are considered for locally conformally flat manifolds.

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1. Introduction

An important problem in differential geometry that is currently being studied is the following:

- (1) Find necessary and sufficient conditions on a symmetric tensor T on a differentiable manifold M^n , $n \geq 3$, such that a metric g can be obtained to satisfy $\text{Ric}_g - \frac{K}{2}g = T$, where Ric_g and K are the Ricci tensor and scalar curvature of the metric g , respectively.

When g is a Lorentzian metric on a four dimensional manifold, the equation $\text{Ric}_g - \frac{\bar{K}}{2}g = T$ is called the Einstein field equation. Studying problem (1) corresponds to solving a system of nonlinear second-order differential equations. When T is non-singular, a local solution to the problem (1) always exists, as shown by DeTurck [4]. When a Riemannian manifold is compact, problem (1) appears as a list of proposed problems of the Seminar on Differential Geometry edited by Yau [12]. For a tensor T that represents a physical field such as an electromagnetic field of perfect fluid, pure radiation field, and vacuum ($T = 0$), the above equation has been studied in several articles, some in which the authors have found solutions that are invariant under certain symmetry groups of the equation (see [6] for details). When the metric g is conformal to the Relativistic spacetime, then the solutions in the vacuum case are flat (see [6]). In the case of pure radiation, perfect fluid and electromagnetic field all solutions that are conformal to the Relativistic spacetime are known. We refer to [8] for the pure radiation or the null electromagnetic field, [6, 7] for the non-null electromagnetic field, and [1, 6] if T is a perfect fluid. In addition to these results, as far as we know, there isn't many studies on the problem (1) with respect to other manifolds, its dimensions, or the tensor T .

Pina and Tenenblat obtained results for problem (1) for special classes of tensors T in locally conformally flat manifolds (see [9–11] and their references).

In this work, we consider a diagonal $(0,2)$ -tensor T on the pseudo-Euclidean space $(\mathbb{R}^n, g)$, $n \geq 3$ and provide necessary as well as sufficient conditions for the existence of a metric conformal to g , whose Einstein tensor is the given tensor T . Moreover, we extend the theory to locally conformally flat manifolds.

Using the obtained results, we will construct solutions to the equation of the static perfect fluid spacetime.

2. Preliminaries

We consider the pseudo-Euclidean space $(\mathbb{R}^n, g)$, $n \geq 3$, with coordinates $x = (x_1, \dots, x_n)$ and $g_{ij} = \delta_{ij}\epsilon_i$, $1 \leq i, j \leq n$, where δ_{ij} is the delta Kronecker, $\epsilon_i = \pm 1$, with at least one $\epsilon_i = 1$ and diagonal $(0,2)$ -tensor of the form $T = \sum_i \epsilon_i h_i(x) dx_i^2$, with $h_i(x)$ smooth functions. We seek necessary and sufficient

conditions on tensor T for the existence of metric $\bar{g} = \frac{1}{\psi^2}g$ such that $\text{Ric}_{\bar{g}} - \frac{\bar{K}}{2}\bar{g} = T$. That is, given the tensor T , we wish to solve the problem

$$\begin{cases} \bar{g} = \frac{1}{\psi^2}g \\ \text{Ric}_{\bar{g}} - \frac{\bar{K}}{2}\bar{g} = T \end{cases} \quad (2.1)$$

for the function ψ . Since $\bar{g}$ is conformal to the pseudo-Euclidean metric g , the Ricci tensor of $\bar{g}$ is given by

$$\text{Ric}_{\bar{g}} = \frac{1}{\psi^2} \left\{ (n-2)\psi \text{Hess}_g \psi + (\psi \Delta_g \psi - (n-1)\|\nabla_g \psi\|^2) g \right\} \quad (2.2)$$

and the scalar curvature of $\bar{g}$ is given by

$$\bar{K} = (n-1) (2\psi \Delta_g \psi - n\|\nabla_g \psi\|^2), \quad (2.3)$$

where $\text{Hess}_g \psi$ is the Hessian of the function ψ with respect to the pseudo-Euclidean metric, $\Delta_g \psi$ denotes the Laplacian, and $\nabla_g \psi$ the gradient in the pseudo-Euclidean metric g (see [2]).

The Einstein tensor of metric $\bar{g}$ can be expressed by

$$\begin{aligned} \text{Ric}_{\bar{g}} - \frac{\bar{K}}{2} \bar{g} &= \frac{1}{\psi^2} \left\{ (n-2)\psi \text{Hess}_g \psi + \left(-(n-2)\psi \Delta_g \psi + \frac{(n-1)(n-2)}{2} \|\nabla_g \psi\|^2 \right) g \right\} \\ &= (n-2) \left\{ \frac{\text{Hess}_g \psi}{\psi} + \left(-\frac{\Delta_g \psi}{\psi} + (n-1) \frac{\|\nabla_g \psi\|^2}{2\psi^2} \right) g \right\}. \end{aligned} \quad (2.4)$$

We denote by $\psi_{,x_k}$ and h_{i,x_k} the partial derivatives of ψ and h_i with respect to x_k , respectively, and $\psi_{,x_i x_j}$ and $h_{i,x_i x_j}$ the second-order derivatives of ψ and h_i with respect to $x_i x_j$, respectively. As g is the pseudo-Euclidean metric on $\mathbb{R}^n$, $n \geq 3$, studying problem (2.1) when $T = \sum_i \epsilon_i h_i(x) dx_i^2$, $h_i(x)$ are smooth functions is equivalent to studying the following system of equations:

$$\begin{cases} (n-2) \left(\frac{\psi_{,x_i x_i}}{\psi} - \epsilon_i \frac{\Delta_g \psi}{\psi} + \epsilon_i (n-1) \frac{\|\nabla_g \psi\|^2}{2\psi^2} \right) = \epsilon_i h_i, \forall i. \\ \psi_{,x_i x_j} = 0, \quad \forall \quad i \neq j. \end{cases} \quad (2.5)$$

From the second equation of (2.5), it follows that ψ can be expressed as a sum of functions, each of which depends only on one of the variables x_i ; therefore, we write $\psi(x) = \sum_{i=1}^n \psi_i(x_i)$.

Set $\Lambda = \sum_{k=1}^n h_k$. We will study the system (2.5) with the additional condition that $\Lambda(x) \neq \frac{n-1}{2}(h_i(x) + h_j(x))$, for all $x \in \mathbb{R}^n$ and all $i \neq j$.

The condition in $\Lambda(x) \neq \frac{n-1}{2}(h_i(x) + h_j(x))$, for all $x \in \mathbb{R}^n$ and all $i \neq j$ is algebraic, being fundamental to guarantee that the equations in Theorem 3.1 are well defined.

3. Main Results

In this section, we present our main results. The proofs will be provided in the following section.

The next theorem is our main result that contains necessary and sufficient conditions for problem (2.1) such that a solution can be obtained.

Theorem 3.1. *Let $(\mathbb{R}^n, g)$, $n \geq 3$ be the pseudo-Euclidean space with coordinates $x_1, \dots, x_n$ and metric $g_{ij} = \delta_{ij}\epsilon_i$. Consider the diagonal $(0, 2)$ -tensor*

$$T = \sum_i^n \epsilon_i h_i(x) dx_i^2, \text{ where } h_i(x) \text{ are smooth functions such that}$$

$\Lambda(x) \neq \frac{n-1}{2}(h_i(x) + h_j(x))$ for all $x \in \mathbb{R}^n$ and all $i \neq j$. A positive function ψ exists such that the metric $\bar{g} = \frac{1}{\psi^2}g$ satisfies $\text{Ric}_{\bar{g}} - \frac{\bar{K}}{2}\bar{g} = T$ if and only if the functions h_i satisfy the following system of differential equations:

$$\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} = \frac{\alpha_{k,x_j}}{\alpha_k + \alpha_j}, \quad i \neq j, \quad k \neq j, \quad (3.1)$$

$$\left(\frac{\alpha_{j,x_i}}{\alpha_i + \alpha_j} \right)_{,x_k} = \left(\frac{\alpha_{j,x_k}}{\alpha_j + \alpha_k} \right)_{,x_i}, \quad i \neq j, \quad k \neq j, \quad (3.2)$$

$$\left(\frac{\alpha_{j,x_i}}{\alpha_i + \alpha_j} \right)_{,x_j} = \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_i}, \quad i \neq j, \quad (3.3)$$

$$\begin{aligned} & \frac{n-1}{2} \epsilon_i \left(\frac{\alpha_{j,x_i}}{\alpha_i + \alpha_j} \right)^2 + \sum_{k \neq i} \epsilon_k \left(\frac{\alpha_{j,x_k}}{\alpha_j + \alpha_k} \right)_{,x_k} \\ & + \frac{(n-3)}{2} \sum_{k \neq i} \epsilon_k \left(\frac{\alpha_{j,x_k}}{\alpha_j + \alpha_k} \right)^2 = \frac{h_i}{n-2}, \quad i \neq j \end{aligned} \quad (3.4)$$

$$\frac{\alpha_{j,x_i} \alpha_{i,x_j}}{(\alpha_i + \alpha_j)^2} = \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_i}, \quad i \neq j. \quad (3.5)$$

where $\alpha_i = \frac{1}{(n-2)} \left(h_i - \frac{\Lambda}{(n-1)} \right)$. If the functions h_i satisfy these conditions up to a multiplicative constant, then

$$\psi(x) = \exp \left(- \int \frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} dx_j + \beta_j(\hat{x}_j) \right), \quad (3.6)$$

where the variable x_j is arbitrary and $\beta(\hat{x}_j)$ does not depend on the variable x_j , being $\hat{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ and $\beta(\hat{x}_j)$ satisfies the following system of $(n-1)$ differential equations

$$\beta_{j,x_i} = \int \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_i} dx_j - \frac{\alpha_{j,x_i}}{\alpha_i + \alpha_j}, \quad i \neq j. \quad (3.7)$$

The case for $T = hg$, where h is a nonvanishing smooth function, has been studied in [10].

Remark 3.2. Consider that the functions h_i given in the Theorem 3.1 are defined in W , where W is an open subset of $\mathbb{R}^n$ and that $\Lambda(x) \neq \frac{n-1}{2}(h_i(x) +$

$h_j(x))$, $\forall x \in W$ with $i \neq j$. Then all conclusions obtained in the Theorem 3.1 are still valid in W .

Theorem 3.3. (See [10]) Let $(\mathbb{R}^n, g)$, $n \geq 3$, be a pseudo-Euclidean space, with coordinates $x_1, \dots, x_n$ and metric $g_{ij} = \delta_{ij}\epsilon_i$. A metric $\bar{g} = \frac{1}{\psi^2}g$ exists such that $\text{Ric}_{\bar{g}} - \frac{\bar{K}}{2}\bar{g} = hg$, where h is a nonvanishing smooth function if and only if

$$h(x) = \frac{(n-1)(n-2)\gamma}{2\psi^2} \quad (3.8)$$

where a, b_i, c are constants, $\gamma = \sum_{k=1}^n \epsilon_k b_k^2 - 4ac$ and

$$\psi(x) = \sum_{i=1}^n (\epsilon_i a x_i^2 + b_i x_i) + c. \quad (3.9)$$

The metric $\bar{g}$ is unique up to homothety. Moreover, when g is a Euclidean metric, we have

- 1 if $\gamma < 0$, then $\bar{g}$ is globally defined on $\mathbb{R}^n$ and T is negative definite;
- 2 if $\gamma \geq 0$, then excluding the homothety, the set of singularity points of $\bar{g}$ comprises
 - a) a point if $\gamma = 0$;
 - b) a hyperplane if $\gamma > 0$ and $a = 0$;
 - c) an $(n-1)$ -dimensional sphere if $\gamma > 0$ and $a \neq 0$.

Remark 3.4. Equations (3.1) to (3.5) of Theorem 3.1 that are satisfied with $h_i = h$, $i = 1, \dots, n$, and ψ as given by (3.9) are the same as those in Theorem 3.1. This demonstrates that examples of tensors exist in $\mathbb{R}^n$, which are solutions of the equations of Theorem 3.1.

We present the result of the non-existence of conformal metrics to a special type of tensor T .

Theorem 3.5. Let $(\mathbb{R}^n, g)$, $n \geq 3$ be a pseudo-Euclidean space, with coordinates $x_1, \dots, x_n$ and metric $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Consider the tensor $T = \sum_{i=1}^n \epsilon_i h_i(x_i) dx_i^2$, where the functions h_i only depend on the variable x_i and $(n-1)(h_i + h_j) - 2\Lambda \neq 0$ for all $i \neq j$. Then, there is no metric $\bar{g} = g/\psi^2$, such that $\text{Ric}_{\bar{g}} - \frac{\bar{K}}{2}\bar{g} = T$.

In the particular case in which the components of the tensor T depend only on one variable, we have the following result.

Theorem 3.6. Let $(\mathbb{R}^n, g)$, $n \geq 3$ be a pseudo-Euclidean space, with coordinates $x_1, \dots, x_n$ and metric $g_{ij} = \delta_{ij}\epsilon_i$. Consider a diagonal $(0, 2)$ -tensor $T = \sum_{i=1}^n \epsilon_i h_i(x_k) dx_i^2$, with $h_i(x_k)$ nonvanishing smooth functions that depend only on x_k for some fixed k , $1 \leq k \leq n$, such that $\Lambda(x) \neq \frac{(n-1)}{2}(h_i(x) + h_j(x))$ for all $x \in \mathbb{R}^n$ and all $i \neq j$. There exists a metric $\bar{g} = g/\psi^2$ such that $\text{Ric}_{\bar{g}} - (\bar{K}/2)\bar{g} = T$ if and only if all the functions h_i for $i \neq k$ are equal to

one another, e.g., $h_i = h$ for $i \neq k$, and the functions h and h_k satisfy the system

$$\frac{(3-n)}{2} \left(\frac{h_{k,x_k}}{(n-3)h_k - (n-1)h} \right)^2 + \left(\frac{h_{k,x_k}}{(n-3)h_k - (n-1)h} \right)_{,x_k} = -\frac{1}{(n-2)} \epsilon_k h \quad (3.10)$$

and

$$\left(\frac{h_{k,x_k}}{(n-3)h_k - (n-1)h} \right)^2 = \frac{2}{(n-1)(n-2)} \epsilon_k h_k \quad (3.11)$$

with $(n-3)h_k - (n-1)h \neq 0$. If h_k and h satisfy these conditions, then ψ depends only on x_k and ψ is given by

$$\psi(x_k) = C \exp \left(\int \frac{h_{k,x_k}}{(n-3)h_k - (n-1)h} dx_k \right), \quad (3.12)$$

where C is a positive constant.

Corollary 3.7. Let $(\mathbb{R}^{n+1}, g)$, $n \geq 2$ be a pseudo-Euclidean space, with the diagonal $(0, 2)$ -tensor

$$T = h_k(x_k) \epsilon_k dx_k^2 + h(x_k) \sum_{i \neq k} \epsilon_i dx_i^2,$$

where

$$h_k(x_k) = \frac{(n-1)(n-2)\epsilon_k \xi^2}{2} \quad \text{and} \quad h(x_k) = \frac{(n-2)(n-3)\epsilon_k \xi^2}{2} + (n-2)\epsilon_k \xi_{,x_k},$$

with $\xi = \xi(x_k)$ a smooth function that depends only on x_k for the same fixed k , $1 \leq k \leq n$. Then, a conformal metric $\tilde{g} = g/\psi^2$ exists such that $\text{Ric}_{\tilde{g}} - \frac{\tilde{K}}{2} \tilde{g} = T$ and

$$\psi(x_k) = C \exp \left(- \int \xi(x_k) dx_k \right), \quad (3.13)$$

where C is a positive constant.

Remark 3.8. Consider $(\mathbb{R}^{n+1}, g)$, $n \geq 2$ as a Euclidean space in Corollary 3.7. If, in addition, $0 \leq \left| \int \xi(x_k) dx_k \right| \leq L$ for a finite constant L , then $\bar{g}$ is complete on $\mathbb{R}^{n+1}$ (see [11]).

We can extend Theorem 3.1 to locally conformally flat manifolds.

Theorem 3.9. Let (M^n, g) , $n \geq 3$, be a locally conformally flat Riemannian manifold. Let V be an open subset of M with cartesian coordinates $x = (x_1, \dots, x_n)$ where $g_{ij} = \frac{\delta_{ij}}{F^2}$. Consider the diagonal $(0, 2)$ -tensor $T = \sum_{i=1}^n h_i(x) dx_i^2$, where h_i are non-vanishing smooth functions such that $\Lambda(x) \neq \frac{(n-1)}{2} (h_i(x) + h_j(x))$ for all $x \in V$ and all $i \neq j$. There exists a metric $\bar{g} = g/\phi^2$ on V such that $\text{Ric}_{\bar{g}} - \frac{\bar{K}}{2} \bar{g} = T$ if, and only if, the functions h_i satisfy the conditions of Theorem 3.1, in which case ψ is given by Eq.(3.6) and then $\phi = \frac{\psi}{F}$.

Remark 3.10. Analogously, we can extend Theorem 3.6 to locally conformally flat manifolds.

As an application of Theorem 3.9, we demonstrate that given a certain $(0, 2)$ -tensor T in $\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}, x_n > 0\}$, a complete metric $\bar{g}$ exists, conformal to the metric of the hyperbolic space whose Einstein tensor is T .

Example 3.11. Consider the semi-space of $\mathbb{R}^n$ given by

$$\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}, x_n > 0\} \quad (3.14)$$

Insert in $\mathbb{R}_+^n$ the metric

$$g_{ij} = \frac{\delta_{ij}}{x_n^2}. \quad (3.15)$$

Given the diagonal $(0, 2)$ -tensor

$$T = h_n(x_n)dx_n^2 + h(x_n) \sum_{i \neq n} dx_i^2 \quad (3.16)$$

on $\mathbb{H}^n = (\mathbb{R}_+^n, g)$, where

$$h_n(x_n) = \frac{(n-2)(n-1)}{2x_n^2} (2x_n^2 + x_n K - 1)^2,$$

and

$$h(x_n) = \frac{(n-2)((n-3)(2x_n^2 + Kx_n - 1)^2 + 2(2x_n^2 + 1))}{2x_n^2}$$

$K \in \mathbb{R}_+$, by Theorem 3.9, the metric $\bar{g} = g/\phi^2$ exists, where $\phi = \psi/F = e^{-(1+K)x_n^2}$, such that $\text{Ric}_{\bar{g}} - \frac{K}{2}\bar{g} = T$. Moreover, because $\phi(x)$ is bounded, $(\mathbb{R}_+^n, \bar{g})$ is a complete Riemannian manifold that is conformal to the hyperbolic space.

As an application of Corollary 3.7, we construct an example of a static perfect fluid spacetime. Prior to this application, we present a brief introduction regarding the static manifolds and perfect fluids.

4. Static Perfect Fluid

A semi-Riemannian $(n+1)$ -manifold $(\tilde{M}^{n+1}, \tilde{g})$ is called (globally) static if $(\tilde{M}, \tilde{g})$ can decompose into a warped product of the following type:

$$(\tilde{M}^{n+1}, \tilde{g}) = (\bar{M}^n, \bar{g}) \times_f (\mathbb{R}, -dt^2) \quad (4.1)$$

and the metric $\tilde{g}$ has the form

$$\tilde{g} = x^* \bar{g} - f(x)^2 dt^2, \quad (4.2)$$

where $t : \tilde{M} \rightarrow \mathbb{R}$ and $x : \tilde{M} \rightarrow \bar{M}$ are natural projections, $\bar{g}$ is a semi-Riemannian metric on $\bar{M}$, and $f \in C^\infty(\bar{M})$ is strictly positive.

A perfect fluid stress-energy tensor over a Lorentzian $(n+1)$ -manifold may be recognized by the identity

$$T = (\mu + \nu)\tilde{g}(\cdot, X) \otimes \tilde{g}(\cdot, X) + \nu\tilde{g},$$

where $\mu, \nu \in C^\infty(\bar{M})$ are the energy density and pressure, respectively; $X \in \mathfrak{X}(\bar{M})$ is vector field such that $\tilde{g}(X, X) = -1$ and represents the flux of the fluid (cf. [5]). In our study, we consider the perfect fluid stress-energy tensor for which $X = f(x)^{-1}\partial_t$, thus, given that $\tilde{g}(\cdot, X) = -f(x)dt$, T can be recognized by

$$T = \nu x^* \bar{g} + \mu f(x)^2 dt^2. \quad (4.3)$$

Definition 4.1. A semi-Riemannian $(n+1)$ -dimensional $(\bar{M}^{n+1}, \tilde{g})$ is called static fluid perfect spacetime if $(\bar{M}, \tilde{g})$ is static and the metrics $\tilde{g}$ satisfies Einstein's equation $Ric_{\tilde{g}} - \frac{\bar{K}}{2}\tilde{g} = T$, where T is a perfect fluid stress-energy tensor is given by $T = \nu x^* \bar{g} + \mu f(x)^2 dt^2$ and μ, ν are the energy density and pressure, respectively.

Therefore, from the previous definition, we know that the static perfect fluid equation is equivalent to

$$Ric_{\tilde{g}} - \frac{Hess_{\tilde{g}}f}{f} = \frac{1}{n} \left(\bar{K} - \frac{\Delta_{\tilde{g}}f}{f} \right) \tilde{g}. \quad (4.4)$$

Hence, the following relations exist:

$$\mu = \frac{\bar{K}}{2}, \quad \nu = \frac{n-1}{n} \left(\frac{\Delta_{\tilde{g}}f}{f} - \frac{n-2}{2(n-1)} \bar{K} \right). \quad (4.5)$$

For more details, see [3].

The next theorem provides necessary and sufficient conditions such that the product manifold $(M^n, \bar{g}) \times_f \mathbb{R}$ obtained in Corollary 3.7 is a static perfect fluid spacetime. Without loss of generality in Corollary 3.7, we consider $h = h(x_k)$ a smooth function that depends only on x_k , for the same fixed k , $1 \leq k \leq n$. A conformal metric $\tilde{g} = g/\psi^2$ exists such that $Ric_{\tilde{g}} - \frac{\bar{K}}{2}\tilde{g} = T$, and $\psi(x_k) = C \exp(-\int h(x_k) dx_k)$, where C is a positive constant and g is a pseudo-Euclidean metric given by

$$g = \sum_{i=1}^n \epsilon_i dx_i \otimes dx_i. \quad (4.6)$$

Using $\epsilon_{n+1} = -1$ in $\tilde{g}$, we obtain

$$\tilde{g} = \frac{g}{\psi^2(x_k)} - f(x_k)^2 dx_{n+1}^2, \quad (4.7)$$

where $f = \frac{1}{\psi}$, consequently,

$$(\mathbb{R}^{n+1}, \tilde{g}) = \left(\mathbb{R}^n, \bar{g} = \frac{1}{\psi^2(x_k)} \epsilon_i \delta_{ij} \right) \times_f (\mathbb{R}, -dx_{n+1}^2), \quad 1 \leq i, j \leq n, \quad (4.8)$$

is a static manifold.

Theorem 4.2. Let $(\mathbb{R}^{n+1}, \tilde{g}) = \left(\mathbb{R}^n, \bar{g} = \frac{1}{\psi(x_k)^2} \epsilon_i \delta_{ij}\right) \times_f (\mathbb{R}, -dx_{n+1}^2)$, $1 \leq i, j \leq n$, be a static manifold. Given $\psi = \psi(x_k)$ and $f = \frac{1}{\psi(x_k)}$, the manifold $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, furnished with metric tensor

$$\tilde{g} = \frac{g}{\psi(x_k)^2} - f(x_k)^2 dx_{n+1}^2 \quad (4.9)$$

is a static perfect fluid spacetime if, and only if,

$$\psi(x_k) = a_k x_k + b_k. \quad (4.10)$$

Additionally, μ, ν are given by

$$\mu = -\frac{(n-1)n}{2} \epsilon_k a_k^2, \quad (4.11)$$

and

$$\nu = \frac{(n-1)n}{2} \epsilon_k a_k^2, \quad (4.12)$$

where $a_k \in \mathbb{R}$, and tensor T is given by

$$T = \nu x^* \bar{g} + \mu f(x)^2 dt^2. \quad (4.13)$$

Corollary 4.3. Let $(\mathbb{R}^{n+1}, g), n \geq 2$ be a pseudo-Euclidean space, with the diagonal $(0, 2)$ -tensor

$$T = h_k(x_k) \epsilon_k dx_k^2 + h(x_k) \sum_{i \neq k} \epsilon_i dx_i^2,$$

where

$$h_k(x_k) = \frac{(n-1)(n-2)\epsilon_k \xi^2}{2} \text{ and } h(x_k) = \frac{(n-2)(n-3)\epsilon_k \xi^2}{2} + (n-2)\epsilon_k \xi_{,x_k}$$

with $\xi = \xi(x_k)$ a smooth function that depends only on x_k , for same fixed k , $1 \leq k \leq n$. Then, product manifold $(M^n, \bar{g}) \times_f \mathbb{R}$ is a static perfect fluid spacetime if, and only if, $\xi(x_k) = -\frac{1}{x_k + K}$.

As an application of Corollary 4.3, we construct an example of a static perfect fluid spacetime.

Example 4.4. Without loss of generality, letting $k = n$ and $K = 0$ on Corollary 4.3, we obtain $\xi(x_n) = -\frac{1}{x_n}$, that is, h is defined in $\bar{M} = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$. Consequently, ψ is given by

$$\psi(x_n) = C e^{K_1} x_n, \quad (4.14)$$

where $K_1 \in \mathbb{R}_*^+$ is a constant of integration. Consider the following case:

$$\epsilon_1 = \dots = \epsilon_n = 1, C e^{K_1} = 1 \text{ and } x_n > 0,$$

which corresponds to the n -dimensional hyperbolic space

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\},$$

with metric tensor

$$\bar{g}_{ij} = \frac{\delta_{ij}}{x_n^2},$$

and

$$\psi(x_n) = x_n, \quad f(x_n) = \frac{1}{x_n}.$$

It is noteworthy that

$$(\tilde{M}^{n+1}, \tilde{g}) = (\mathbb{H}^n, \bar{g} = \frac{\delta}{x_n^2}) \times_f (\mathbb{R}, -dx_{n+1}^2), \quad (4.15)$$

is a static perfect fluid spacetime.

5. Proof of the Main Results

Initially, we will prove a lemma that will be highly useful in the proof of the main results.

Lemma 5.1. *Let $\psi(x_1, \dots, x_n)$ be a solution of (2.5). Then,*

$$\frac{\psi, x_j}{\psi} = -\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j}, \quad \forall i \neq j \quad (5.1)$$

and for a distinct i, j, k

$$\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} = \frac{\alpha_{k,x_j}}{\alpha_i + \alpha_j}, \quad (5.2)$$

where $\alpha_i = \frac{1}{n-2} \left(h_i - \frac{\Lambda}{(n-1)} \right)$.

Proof (of Lemma 5.1). Assume that the functions h_i satisfy

$$\Lambda(x) = \sum_{k=1}^n h_k(x) \neq \frac{(n-1)}{2} (h_i(x) + h_j(x)) \text{ for all } x \in \mathbb{R}^n \text{ and all } i \neq j.$$

Since the function ψ is a solution of (2.5), if we sum the first equation in (2.5) on i , we obtain

$$\frac{\Delta\psi}{\psi} = \frac{n \|\nabla\psi\|^2}{2\psi^2} - \frac{\Lambda(x)}{(n-1)(n-2)}.$$

Then, we replace the above equation into (2.5) to obtain

$$\frac{\psi, x_i x_i}{\psi} = \frac{\epsilon_i}{n-2} \left(h_i - \frac{\Lambda(x)}{(n-1)} \right) + \epsilon_i \frac{\|\nabla\psi\|^2}{2\psi^2}.$$

We rewrite this as

$$\epsilon_i \psi, x_i x_i = \alpha_i \psi + \frac{\|\nabla\psi\|^2}{2\psi}, \quad (5.3)$$

where

$$\alpha_i = \frac{1}{n-2} \left(h_i - \frac{\Lambda(x)}{(n-1)} \right). \quad (5.4)$$

Differentiating (5.3) with respect to x_j , $j \neq i$ and recalling that $\psi_{,x_i x_j} = 0$ from the second equation of (2.5), we obtain

$$0 = \psi_{,x_j} \alpha_i + \psi \alpha_{i,x_j} + \frac{1}{2} \left(\sum_{k=1}^n \varepsilon_k \frac{(\psi_{,x_k})^2}{\psi} \right)_{,x_j},$$

equivalently,

$$0 = \psi_{,x_j} \alpha_i + \psi \alpha_{i,x_j} + \left(\varepsilon_j \frac{\psi_{,x_j x_j}}{\psi} - \frac{1}{2\psi^2} \sum_{k=1}^n \varepsilon_k (\psi_{,x_k})^2 \right) \psi_{,x_j}. \quad (5.5)$$

Substituting the equation

$$\varepsilon_j \frac{\psi_{,x_j x_j}}{\psi} = \alpha_j + \frac{1}{2\psi^2} \sum_{k=1}^n \varepsilon_k (\psi_{,x_k})^2$$

into (5.5), we get

$$0 = \psi \alpha_{i,x_j} + (\alpha_i + \alpha_j) \psi_{,x_j}.$$

The condition $\alpha_i + \alpha_j \neq 0$ is equivalent to the assumption $\Lambda \neq \frac{(n-1)}{2}(h_i + h_j)$ on the components of tensor T , so, we conclude that

$$\frac{\psi_{,x_j}}{\psi} = -\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j}.$$

The above equation shows that, for a fixed j , the right side is independent of the index i provided that $i \neq j$. Therefore,

$$\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} = \frac{\alpha_{k,x_j}}{\alpha_k + \alpha_j}, \text{ for } i, j, k \text{ distinct,}$$

and the proof is completed. $\square$

Proof (of Theorem 3.1). Suppose that $\bar{g} = g/\psi^2$ is a solution of $\text{Ric}_{\bar{g}} - \frac{\bar{K}}{2}\bar{g} = T$. From Lemma 5.1, the equation (3.1) is satisfied, and we obtain

$$\psi(x) = \exp \left(- \int \frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} dx_j + \beta^j(\hat{x}_j) \right), \quad (5.6)$$

where the function $\beta^j(\hat{x}_j)$ does not depend on x_j .

We demonstrate that the expression of ψ is independent of the variable of integration, and the function β^j is well defined. Taking the derivative of β^j with respect to x_k , $k \neq j$, and $k \neq i$, where i, j are fixed, we obtain

$$\beta^j_{,x_k} = \int \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_k} dx_j - \frac{\alpha_{j,x_k}}{\alpha_j + \alpha_k}.$$

The first derivative of this expression with respect to x_l , $l \neq j$, and $l \neq i$ yields

$$\beta^j_{,x_k x_l} = \int \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_k x_l} dx_j - \left(\frac{\alpha_{j,x_k}}{\alpha_j + \alpha_k} \right)_{,x_l}.$$

Similarly, we obtain

$$\beta_{,x_l x_k}^j = \int \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_l x_k} dx_j - \left(\frac{\alpha_{j,x_l}}{\alpha_i + \alpha_l} \right)_{,x_k}.$$

Thus, $\beta_{,x_k x_l}^j = \beta_{,x_l x_k}^j$ if, only if ,

$$\left(\frac{\alpha_{j,x_k}}{\alpha_j + \alpha_k} \right)_{,x_l} = \left(\frac{\alpha_{j,x_l}}{\alpha_j + \alpha_l} \right)_{,x_k}, \quad l \neq j, l \neq i. \quad (5.7)$$

Analogously for $k \neq j$ and $i \neq j$, we obtain

$$\beta_{,x_k x_i}^j = \int \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_k x_i} dx_j - \left(\frac{\alpha_{j,x_k}}{\alpha_j + \alpha_k} \right)_{,x_i}$$

and

$$\beta_{,x_i x_k}^j = \int \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_i x_k} dx_j - \left(\frac{\alpha_{j,x_i}}{\alpha_j + \alpha_i} \right)_{,x_k}$$

Therefore, $\beta_{,x_k x_i}^j = \beta_{,x_i x_k}^j$ if, and only if,

$$\left(\frac{\alpha_{j,x_k}}{\alpha_j + \alpha_k} \right)_{,x_i} = \left(\frac{\alpha_{j,x_i}}{\alpha_j + \alpha_i} \right)_{,x_k}. \quad (5.8)$$

Thus, from (5.7) and (5.8), the equation (3.2) is satisfied. Next, taking the derivative of $\beta_{,x_i}^j$ with respect to x_j , $j \neq i$, we obtain

$$\beta_{,x_i x_j}^j = \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_i} - \left(\frac{\alpha_{j,x_i}}{\alpha_j + \alpha_i} \right)_{,x_j}. \quad (5.9)$$

Because $\beta_{,x_i x_j}^j = \beta_{,x_j x_i}^j = 0$,

$$\left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_i} = \left(\frac{\alpha_{j,x_i}}{\alpha_j + \alpha_i} \right)_{,x_j} \quad (5.10)$$

Therefore, equation (3.3) is satisfied.

By the first equation of Lemma 5.1, we have

$$\ln \psi(x) = - \int \frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} dx_j + \beta^j(\widehat{x}_j)$$

Integrating the first equation of Lemma 5.1 with respect to x_s , $s \neq j$, we obtain

$$\ln \tilde{\psi}(x) = - \int \frac{\alpha_{i,x_s}}{\alpha_i + \alpha_s} dx_j + \tilde{\beta}^s(\widehat{x}_s).$$

Setting $P = \ln \psi(x) - \ln \tilde{\psi}(x)$, we state that P is constant. In fact, taking the derivative with respect to x_s , x_j and using the expressions for the derivatives of β^j and $\tilde{\beta}^s$, we obtain

$$P_{,x_s} = - \int \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_s} dx_j + \frac{\alpha_{i,x_s}}{\alpha_i + \alpha_s} + \beta_{,x_s}^j$$

$$\begin{aligned}
&= - \int \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_s} dx_j + \frac{\alpha_{i,x_s}}{\alpha_i + \alpha_s} + \int \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_s} dx_j - \frac{\alpha_{j,x_s}}{\alpha_j + \alpha_s} = 0. \\
P_{,x_j} &= - \frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} + \int \left(\frac{\alpha_{i,x_s}}{\alpha_i + \alpha_s} \right)_{,x_j} dx_s - \tilde{\beta}^s_{,x_j} \\
&= - \frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} + \int \left(\frac{\alpha_{i,x_s}}{\alpha_i + \alpha_s} \right)_{,x_j} dx_s - \int \left(\frac{\alpha_{i,x_s}}{\alpha_i + \alpha_s} \right)_{,x_j} dx_s + \frac{\alpha_{s,x_j}}{\alpha_s + \alpha_j} = 0.
\end{aligned}$$

For $k \neq s$ and $k \neq j$, by the fact that β^j and $\tilde{\beta}^s$ are solutions to (2.5) and using the derivatives of β^j and $\tilde{\beta}^s$, we obtain

$$\begin{aligned}
P_{,x_k} &= - \int \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_k} dx_j + \int \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_k} dx_j - \frac{\alpha_{j,x_k}}{\alpha_j + \alpha_k} \\
&\quad + \int \left(\frac{\alpha_{i,x_s}}{\alpha_i + \alpha_s} \right)_{,x_k} dx_s - \int \left(\frac{\alpha_{i,x_s}}{\alpha_i + \alpha_s} \right)_{,x_k} dx_s + \frac{\alpha_{s,x_k}}{\alpha_s + \alpha_k} \\
&= \frac{\alpha_{s,x_k}}{\alpha_s + \alpha_k} - \frac{\alpha_{j,x_k}}{\alpha_j + \alpha_k} = 0.
\end{aligned}$$

Thus, P is constant and ψ is well defined. Since ψ is a solution of (2.1), then ψ satisfies (2.5). From Lemma 5.1, for $i \neq j$, we have $\psi_{,x_i} = -\psi \frac{\alpha_{j,x_i}}{\alpha_i + \alpha_j}$. Taking the derivative with respect to x_i , we obtain

$$\frac{\psi_{,x_i x_i}}{\psi} = \left(\frac{\alpha_{j,x_i}}{\alpha_i + \alpha_j} \right)^2 - \left(\frac{\alpha_{j,x_i}}{\alpha_i + \alpha_j} \right)_{,x_i}. \quad (5.11)$$

Using $\|\nabla\psi\|^2 = \sum_{k \neq i} \epsilon_k (\psi_{,x_k})^2 + \epsilon_i (\psi_{,x_i})^2$ and the above expression, the first equation in (2.5) is equivalent to

$$\begin{aligned}
&\frac{(n-1)}{2} \left(\frac{\alpha_{j,x_i}}{\alpha_i + \alpha_j} \right)^2 + \epsilon_i \sum_{k \neq i} \epsilon_k \left(\frac{\alpha_{j,x_k}}{\alpha_j + \alpha_k} \right)_{,x_k} + \frac{(n-3)}{2} \epsilon_i \sum_{k \neq j} \epsilon_k \left(\frac{\alpha_{j,x_i}}{\alpha_i + \alpha_j} \right)^2 \\
&= \frac{\epsilon_i}{(n-2)} h_i.
\end{aligned}$$

Simplifying this expression, we obtain

$$\begin{aligned}
&\frac{(n-1)}{2} \epsilon_i \left(\frac{\alpha_{j,x_i}}{\alpha_i + \alpha_j} \right)^2 + \sum_{k \neq i} \epsilon_k \left(\frac{\alpha_{j,x_k}}{\alpha_j + \alpha_k} \right)_{,x_k} + \frac{(n-3)}{2} \sum_{k \neq j} \epsilon_k \left(\frac{\alpha_{j,x_k}}{\alpha_j + \alpha_k} \right)^2 \\
&= \frac{h_i}{(n-2)},
\end{aligned}$$

which proves equation (3.4).

From Lemma 5.1, we obtain for $i \neq j$ that

$$\psi_{,x_j x_i} = -\psi_{,x_i} \frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} - \psi \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_i} = 0.$$

Since $\psi \neq 0$, we have that

$$\frac{\psi_{,x_j x_i}}{\psi} = \frac{\alpha_{j,x_i}}{\alpha_i + \alpha_j} \frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} - \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_i},$$

or equivalently,

$$\frac{\alpha_{j,x_i} \alpha_{i,x_j}}{(\alpha_i + \alpha_j)^2} = \left(\frac{\alpha_{i,x_j}}{\alpha_i + \alpha_j} \right)_{,x_i}.$$

The converse is a straightforward computation. $\square$

Proof (of Theorem 3.5). Since h_i does not depend on variables x_j , $j \neq i$, we have from Lemma 5.1 that

$$\frac{\psi_{,x_j}}{\psi} = \frac{h_{j,x_j}}{(n-1)(h_i + h_j) - 2\Lambda}, \quad \forall j \neq i.$$

By equation (3.1), we obtain

$$h_{j,x_j}(h_k - h_i) = 0, \quad \text{for } i \neq j, \quad j \neq k$$

Therefore, $h_{j,x_j} = 0$ or $h_k - h_i = 0$. If $h_{j,x_j} = 0$, we have $\frac{\psi_{,x_j}}{\psi} = 0$, $\forall j \neq i$, by similar arguments, we have $\frac{\psi_{,x_i}}{\psi} = 0$, $\forall i \neq j$. Thus, $\psi_{,x_k} = 0$ for all k , therefore, ψ is a constant. Using the equation (2.5), we obtain that $h_i = 0$ for all $i = 1, \dots, n$, which contradicts $(n-1)(h_i + h_j) - 2\Lambda \neq 0$, for $i \neq j$. Meanwhile, if $h_k - h_i = 0$, we obtain $h_{k,x_i} = h_{i,x_i} = 0$. Consequently, $\psi_{,x_i} = 0$ for all k ; therefore, ψ is a constant. Therefore, in both case, the metric $\bar{g} = g/\psi^2$ does not exist such that $\text{Ric}_{\bar{g}} - \frac{K}{2} = T$. $\square$

Proof (of Theorem 3.6). Since h_i depends only on x_k from Lemma 5.1, $\psi_{,x_j} = 0$ for every $j \neq k$ and $\psi = \psi(x_k)$. Furthermore, from the first equation in (2.5), we have $h_i = (n-2) \left((n-1) \frac{\|\nabla \psi\|^2}{2\psi^2} - \frac{\Delta \psi}{\psi} \right)$ for all $i \neq k$; therefore, $h_i = h_j = h$ if $i \neq k$ and $j \neq k$. According to Theorem 3.1, the following equations are satisfied:

$$\frac{(3-n)}{2} \left(\frac{h_{k,x_k}}{(n-3)h_k - (n-1)h} \right)^2 + \left(\frac{h_{k,x_k}}{(n-3)h_k - (n-1)h} \right)_{,x_k} = -\frac{1}{(n-2)} \epsilon_k h$$

and

$$\left(\frac{h_{k,x_k}}{(n-3)h_k - (n-1)h} \right)^2 = \frac{2}{(n-1)(n-2)} \epsilon_k h_k.$$

By the first equation of Lemma 5.1, we obtain

$$\frac{\psi_{,x_k}}{\psi} = \frac{h_{k,x_k}}{(n-3)h_k - (n-1)h}.$$

Therefore,

$$\psi(x) = C \exp \left(\int \frac{h_{k,x_k}}{(n-3)h_k - (n-1)h} dx_k \right). \quad (5.12)$$

$\square$

Proof (of Corollary 3.7). Since h_i depends only on x_k from Lemma 5.1, we have $\psi_{,x_j} = 0$ for every $j \neq k$ and $\psi = \psi(x)$. Furthermore, from the first equation in (2.5), we have $h_i = (n-1)(n-2)\frac{||\nabla\psi||}{2\psi^2} - (n-2)\frac{\Delta\psi}{\psi}$ for $i \neq k$; therefore, $h_i = h_j = h$ if $i \neq k$ and $j \neq k$, where

$$\frac{h_{k,x_k}}{(n-3)h_k - (n-1)h} = -\xi.$$

From the expressions of h_k and h , we obtain

$$\frac{(3-n)}{2} \left(\frac{h_{k,x_k}}{(n-3)h_k - (n-1)h} \right)^2 + \left(\frac{h_{k,x_k}}{(n-3)h_k - (n-1)h} \right)_{,x_k} = -\frac{1}{(n-2)}\epsilon_k h.$$

and

$$\left(\frac{h_{k,x_k}}{(n-3)h_k - (n-1)h} \right)^2 = \frac{2}{(n-1)(n-2)}\epsilon_k h_k,$$

therefore, from Theorem 3.6, a conformal metric $\bar{g} = g/\psi^2$ exists such that $Ric_{\bar{g}} - \frac{\bar{K}}{2}\bar{g} = T$ and

$$\psi(x_k) = C \exp \left(- \int \xi(x_k) dx_k \right). \quad (5.13)$$

□

Proof (of Theorem 3.9). Consider $\phi = \frac{\psi}{F}$ and apply Theorem 3.1. □

Proof (of Theorem 4.2). Suppose that the manifold $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ with metric tensor

$$\tilde{g} = g/\psi^2 - f^2 dx_{n+1}^2 \quad (5.14)$$

is a static perfect fluid spacetime; therefore, by equation (4.4), we have

$$f \left(Ric_{\tilde{g}} - \frac{\bar{K}}{n}\tilde{g} \right) = \left(\nabla_{\tilde{g}}^2 f - \frac{\Delta_{\tilde{g}} f}{n}\tilde{g} \right). \quad (5.15)$$

In a coordinate system, we have $\bar{g}_{ij} = 1/\psi^2 \epsilon_i \delta_{ij}$ and

$$\bar{R}_{ij} = \frac{1}{\psi^2} \left\{ (n-2)\psi\psi_{,x_i x_j} + \left(\psi \sum_{k=1}^n \epsilon_k \psi_{,x_k x_k} - (n-1) \sum_{k=1}^n \epsilon_k (\psi_{,x_k})^2 \right) \delta_{ij} \epsilon_i \right\}.$$

Furthermore, as $\psi = \psi(x_k)$ for some k fixed, we get

$$f \left(Ric_{\tilde{g}} - \frac{\bar{K}}{n}\tilde{g} \right)_{ij} = 0, \quad \text{if } i \neq j, \quad (5.16)$$

$$f \left(Ric_{\tilde{g}} - \frac{\bar{K}}{n}\tilde{g} \right)_{ii} = \frac{(2-n)}{n} \epsilon_i \epsilon_k \frac{\psi_{,x_k x_k}}{\psi}, \quad \text{if } i \neq k, \quad (5.17)$$

$$f \left(Ric_{\tilde{g}} - \frac{\bar{K}}{n}\tilde{g} \right)_{kk} = \frac{(n^2 - 3n + 2)}{n} \frac{\psi_{,x_k x_k}}{\psi^2}, \quad \text{if } i = k. \quad (5.18)$$

Similarly, we obtain

$$\nabla_{\bar{g}}^2(f)_{ij} - \frac{\Delta_{\bar{g}}f}{n}\bar{g}_{ij} = 0, \text{ if } i \neq j, \quad (5.19)$$

$$\nabla_{\bar{g}}^2(f)_{ii} - \frac{\Delta_{\bar{g}}f}{n}\bar{g}_{ii} = -\frac{1}{n}\frac{\psi_{,x_kx_k}}{\psi^2}\epsilon_k\epsilon_i, \text{ if } i \neq k, \quad (5.20)$$

$$\nabla_{\bar{g}}^2(f)_{kk} - \frac{\Delta_{\bar{g}}f}{n}\bar{g}_{kk} = \frac{(1-n)}{n}\frac{\psi_{,x_kx_k}}{\psi^2}, \text{ if } i = k. \quad (5.21)$$

Simultaneously, in both cases, we obtain

$$f \left(Ric_{\bar{g}} - \frac{\bar{K}}{n}\bar{g} \right)_{ij} = \left(\nabla_{\bar{g}}^2 f - \frac{\Delta_{\bar{g}}f}{n}\bar{g} \right)_{ij}$$

if, only if, $\psi_{,x_kx_k} = 0$, that is $\psi(x_k) = a_kx_k + b_k$.

For the remaining identities, it is noteworthy that $\psi(x_k) = ax_k + b$, where $a \in \mathbb{R}^*$, $b \in \mathbb{R}$ and $f = \frac{1}{\psi}$; therefore

$$\mu = \frac{n-1}{2}(2\psi\Delta_{\bar{g}}\psi - n||\nabla_{\bar{g}}\psi||^2) = -\frac{(n-1)n}{2}\epsilon_ka^2$$

and

$$\begin{aligned} \nu &= \frac{n-1}{n} \left(\frac{\Delta_{\bar{g}}f}{f} - \frac{n-2}{2(n-1)}\bar{K} \right) \\ &= \frac{n-1}{n} \left(\epsilon_k\psi^3 \left(f_{,x_kx_k} - (n-2)\frac{\psi_{,x_k}f_{,x_k}}{\psi} \right) \right. \\ &\quad \left. - \frac{n-2}{2(n-1)}(n-1)(2\epsilon_k\psi\psi_{,x_kx_k} - n\epsilon_k(\psi_{,x_k})^2) \right) \\ &= \frac{n-1}{n} \left(\epsilon_k\psi^3 \left(\frac{2a^2}{\psi^3} + (n-2)\frac{a^2}{\psi^3} \right) + \frac{(n-2)n}{2}\epsilon_ka^2 \right) \\ &= \frac{(n-1)n}{2}\epsilon_ka^2. \end{aligned}$$

□

Proof (of Corollary 4.3). Using $\epsilon_{n+1} = 1$, the metric obtained in Corollary 3.7 can be written by

$$\tilde{g} = \frac{g}{\psi(x_k)^2} - f(x_k)^2 dx_{n+1}^2, \quad (5.22)$$

where $f(x_k) = \frac{1}{\psi(x_k)}$ and $\psi(x_k) = \text{Cexp}(-\int \xi(x_k)dx_k)$. By Theorem 4.2, T is a static perfect fluid spacetime on $(\mathbb{R}^{n+1}, \tilde{g})$ if, and only if,

$$\psi_{,x_kx_k} = 0,$$

Equivalently, $\xi^2 = \xi_{,x_k}$. Therefore, $\xi = -\frac{1}{x_k + K}$, where $x_k + K \neq 0$. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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Levi Rosa Adriano, Ilton Ferreira de Menezes, Mauricio Donizetti Pieterzack and
Romildo da Silva Pina
Instituto de Matemática e Estatística
Universidade Federal de Goiás
Goiânia 74001-970
Brazil
e-mail: levi@ufg.br;
 iltonmenezesufg@gmail.com;
 mauriciopieterzack@gmail.com;
 romildo@ufg.br

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