



A Result of Rigidity for Ricci-Flat Warped Products

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Abstract

In this work we prove a result of rigidity on Ricci-flat semi-Riemannian warped products when the base is locally conformally flat with scalar curvature zero and invariant by the action of the pseudo-orthogonal group. We obtain that in these manifolds the fiber must be 1-dimensional. As an application, every metric with these characteristics is a generalized Schwarzschild metric.

Keywords Ricci-flat warped product · Generalized Schwarzschild metric · Pseudo-orthogonal group

Mathematics Subject Classification 53C20 · 83C22 · 53C21 · 30F45

1 Introduction and Main Results

In 1915, Einstein proposed a field equation to describe the gravitational effects produced by a given mass in general relativity,

$$Ric_g - \frac{1}{2}sg = T,$$

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where g is a Lorentz metric in a manifold M^4 , s is the scalar curvature of (M^4, g) and T is the stress/energy tensor, a symmetric 2-covariant tensor field in M . This equation is known as the Einstein field equation (Arthur 1987). When $T \equiv 0$, we obtain an equation for a gravitational field in a vacuum and in this case, the metric g satisfies the condition

$$Ric_g = \lambda g, \quad \lambda \in \mathbb{R}. \quad (1.1)$$

Semi-Riemannian manifolds (M^n, g) that satisfy (1.1) are said to be Einstein. In particular, when $\lambda = 0$, (M^n, g) is said to be Ricci-flat.

The classical solution for the Einstein field equation in a vacuum is the Schwarzschild metric

$$ds^2 = \left(1 + \frac{\mu}{2r^{\frac{1}{2}}}\right)^4 (dx_1^2 + dx_2^2 + dx_3^2) - \left(\frac{1 - \frac{\mu}{2r^{\frac{1}{2}}}}{1 + \frac{\mu}{2r^{\frac{1}{2}}}}\right)^2 dt^2, \quad (1.2)$$

where μ is the gravitating mass and $r = x_1^2 + x_2^2 + x_3^2$ (see Einstein 1939). It was found in 1916 by Schwarzschild and models the gravitational fields outside an isolated, static, spherically symmetric star.

An Einstein metric is a good candidate for privileged metric on a given manifold. In dimension two, the only Einstein manifolds we have are spaces forms, which have constant (Gaussian) curvature. In dimensions greater than two, having constant Ricci curvature is a good condition, representing an intermediate level between constant sectional curvature and constant scalar curvature. Einstein manifolds are special in different contexts: they are fixed points of the famous Ricci Flow and Yamabe Flow, critical points of Riemannian functionals, among others. There are many studies on such manifolds and the interest in obtaining new classifications and examples extends to the present day.

Conforming transformations between Einstein spaces were studied in Kühnel (1988). In particular, Kühnel showed that locally conformally flat Einstein manifolds have constant sectional curvature. This result was generalized for the semi-Riemannian case (Kühnel and Rademacher 1995). So, we have no non-trivial Einstein manifolds with metric conformal to a pseudo-Euclidean metric.

A good strategy to obtain locally conformally flat manifolds is to study warped product is a manifold product $B^n \times F^m$ of two semi-Riemannian manifolds (B^n, g_B) and (F^m, g_F) , equipped with the metric $g_B \oplus h^2 g_F$, where $h : B \rightarrow \mathbb{R}_+^*$. The manifold B is said to be base, F is the fiber and h is the warping function, and the warped product is denoted by $B \times_h F$. For a warped product to be locally conformally flat, we need strong restrictions on the base and the fiber (see Brozos-Vázquez et al. 2005).

In Kim and Kim (2003), the authors presented necessary and sufficient conditions for a warped product to be an Einstein manifold. More precisely, a Riemannian warped product $M = B \times_h F$ is Einstein with $Ric_g = \rho g$ if and only if

$$Ric_{g_B} = \rho g_B + \frac{r}{h} Hess_{g_B} h,$$

$$\begin{aligned} Ric_{g_F} &= \mu g_F, \\ h \Delta_{g_B} h + (r-1) |grad_{g_B} h|^2 + \rho h^2 &= \mu, \end{aligned} \quad (1.3)$$

where g_B and g_F represent metrics of the base and fiber, respectively. Then, in an Einstein warped product, the base is a quasi-Einstein manifold and the fiber is an Einstein manifold. This result is valid in the semi-Riemannian case.

There are many studies on Einstein manifolds with warped product metric (see for example Barros et al. 2015; Batista et al. 2020; Case et al. 2011; de Sousa and Pina 2017; Leandro 2018; Rimoldi 2011). Despite this, a complete classification of Einstein warped products is still an open problem.

In this work we prove the following result.

Theorem 1.1 *Every Ricci-flat warped product metric invariant by rotation with base locally conformally flat of scalar curvature zero is a generalized Schwarzschild metric.*

Generalized Schwarzschild metrics invariant by rotation were obtained in dos Santos and Leandro (2019). Santos and Leandro studied the static vacuum Einstein equation and by using an ansatz method, they found the most general invariant that reduces the system of PDEs which describes the problem to a system of ODEs. The invariant found by them shows that the maximum invariance is obtained by rotation. They also provided the entire set of solutions of the reduced system.

In de Sousa and Pina (2017), the authors studied Einstein warped products when the base is conformal to a pseudo-Euclidean space. They presented the system of partial differential equations (PDEs) that describes the problem and obtained all the solutions Ricci-flat invariant under the action of the translation group $(n-1)$ -dimensional.

In this work, unlike what was done in de Sousa and Pina (2017), we seek solutions for a system of PDEs invariant by rotation, i.e., when the functions assume the same value across spheres $(n-1)$ -dimensional in $\mathbb{R}^n$, transforming the system of PDEs into a system of ordinary differential equations (ODEs) (Theorem 1.3). In an attempt to find solutions, we obtain a strong rigidity over the fiber F^m . When $\lambda = 0$ and the scalar curvature of the base is zero, we shall prove that it must be 1-dimensional (Theorem 1.4) and explain all the solutions (Corollary 1.5).

We observe that the metrics found in this work are generalizations of the Schwarzschild metric. We can obtain it by making particular choices for the constants that appear in the solutions given by Corollary 1.5.

Next, we present the results obtained in our study. For a function $\psi : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$, we denote the partial derivatives of first and second orders by $\psi_{,x_i}$ and $\psi_{,x_i x_j}$, respectively.

We start with a result presented in de Sousa and Pina (2017).

Theorem 1.2 *Let $(\mathbb{R}^n, g)$ be a pseudo-Euclidean space, $n > 2$, with coordinates $x = (x_1, \dots, x_n)$ and $g_{ij} = \delta_{ij} \varepsilon_i$, where $\varepsilon_i = \pm 1$. Consider a warped product $M = \mathbb{R}^n \times_h F^m$ with metric $\tilde{g} = \bar{g} + h^2 g_F$, where $\bar{g} = \frac{1}{\psi^2} g$, F is a semi-Riemannian Einstein manifold with constant Ricci curvature λ_F , $m \geq 1$, and $h, \psi : \mathbb{R}^n \rightarrow \mathbb{R}^*$ are smooth functions, where h is a positive function. Then the warped product metric $\tilde{g}$ is*

Einstein with constant Ricci curvature λ if and only if functions h and ψ satisfy:

$$(n-2)h\psi_{,x_i x_j} - m\psi h_{,x_i x_j} - m\psi_{,x_i} h_{,x_j} - m\psi_{,x_j} h_{,x_i} = 0, \quad \forall i \neq j, \quad (1.4)$$

$$(n-2)\psi\psi_{,x_i x_i} h + \varepsilon_i \sum_{k=1}^n \varepsilon_k \psi_{,x_k x_k} - (n-1)\varepsilon_i \sum_{k=1}^n \varepsilon_k (\psi_{,x_k})^2 - \frac{m\psi^2}{h} \left[h_{,x_i x_i} + 2 \frac{\psi_{,x_i} h_{,x_i}}{\psi} - \varepsilon_i \sum_{k=1}^n \varepsilon_k \frac{\psi_{,x_k} h_{,x_k}}{\psi} \right] = \varepsilon_i \lambda, \quad \forall i, \quad (1.5)$$

and

$$\sum_{k=1}^n \varepsilon_k \left[-\psi^2 h h_{,x_k x_k} + (n-2)\psi\psi_{,x_k} h h_{,x_k} - (m-1)\psi^2 h_{,x_k}^2 \right] = \lambda h^2 - \lambda_F. \quad (1.6)$$

Our goal is to determine solutions to the system of Eqs. (1.4), (1.5) and (1.6) invariant by rotation; i.e. solutions of the form $\psi(r)$ and $h(r)$, where $r = \sum_{i=1}^n \varepsilon_i x_i^2$, $r > 0$. The invariant r reduces this system of partial differential equations (PDEs) to a system of ordinary differential equations (ODEs).

Theorem 1.3 *Let $(\mathbb{R}^n, g)$, $n > 2$, be a pseudo-Euclidean space with coordinates $x = (x_1, \dots, x_n)$ and metric components $g_{ij} = \delta_{ij}\varepsilon_i$, where $\varepsilon_i = \pm 1$. Consider a warped product $M = \mathbb{R}^n \times_h F^m$ with metric $\tilde{g} = \bar{g} + h^2 g_F$, where $\bar{g} = \frac{1}{\psi^2} g$, F is a semi-Riemannian Einstein manifold with constant Ricci curvature λ_F , $m \geq 1$, and smooth functions $\psi(r)$ and $h(r)$, where $r = \sum_{k=1}^n \varepsilon_k x_k^2$, $r > 0$. Then $(M, \tilde{g})$ is an Einstein manifold with constant Ricci curvature λ if and only if functions h and ψ satisfy the following system of equations*

$$(n-2)h\psi'' - m\psi h'' - 2m\psi' h' = 0, \quad (1.7)$$

$$\psi \left[2(n-1)\psi' h - m\psi h' \right] + 2r \left[\psi\psi'' h - (n-1)(\psi')^2 h + m\psi\psi' h' \right] = \frac{\lambda}{2} h, \quad (1.8)$$

$$-2n\psi^2 h h' + 4r \left[-\psi^2 h h'' + (n-2)\psi\psi' h h' - (m-1)\psi (h')^2 \right] = \lambda h^2 - \lambda_F. \quad (1.9)$$

Now we prove the main result of this work, which consists of a rigidity for fiber F^m of the warped product described in Theorem 1.3 when the base have scalar curvature $K = 0$ and $\lambda = 0$.

Theorem 1.4 *Let $(\mathbb{R}^n, g)$, $n > 2$, be a pseudo-Euclidean space with coordinates $x = (x_1, \dots, x_n)$ and metric component $g_{ij} = \delta_{ij}\varepsilon_i$, where $\varepsilon_i = \pm 1$. Consider a Ricci-flat warped product*

$$\left(\mathbb{R}^n \times_f F^m, \frac{1}{\psi(r)^2} g \oplus h(r)^2 g_F \right),$$

with $g_{ij} = \delta_{ij}\varepsilon_i$, $\varepsilon_i = \pm 1$, $r = \sum_{i=1}^n \varepsilon_i x_i^2$, $r > 0$, and (F^m, g_F) a semi-Riemannian Einstein manifold. If $\left(\mathbb{R}^n, \frac{1}{\psi(r)^2}\right)$ has scalar curvature $K = 0$, then fiber F is 1-dimensional.

Theorem 1.4 is really notable. It shows us that the real line is the only possibility for the fiber of a Ricci-flat warped product invariant by rotation when the base is locally conformally flat and has scalar curvature zero.

In the next result we obtain all semi-Riemannian Ricci-flat warped product when the base is conformal to the n -dimensional pseudo-Euclidean space with scalar curvature $K = 0$ and fiber F is $(\mathbb{R}, g_0)$, with $g_0 = \pm dt^2$, considering functions h and ψ invariant by rotation.

Corollary 1.5 *Let $(\mathbb{R}^n, g)$, $n > 2$, be a pseudo-Euclidean space with coordinates $x = (x_1, \dots, x_n)$ and metric component $g_{ij} = \delta_{ij}\varepsilon_i$, where $\varepsilon_i = \pm 1$. Consider smooth functions $\psi(r)$ and $h(r)$ with $r = \sum_{k=1}^n \varepsilon_k x_k^2$, $r > 0$. Then*

$$(M, \tilde{g}) = (\mathbb{R}^n \times \mathbb{R}, \bar{g} \oplus h^2 g_0),$$

where $\bar{g} = \frac{1}{\psi^2} g$ and $g_0 = \pm dt^2$, is a Ricci-flat manifold with scalar curvature of base $K = 0$ if and only if functions ψ and h are given by

$$\begin{cases} \psi(r) = k_1 \left(r^{\frac{2-n}{2}} + k \right)^{\frac{2}{2-n}}, \\ h(r) = k_2^{\frac{(n-2)k}{2}} \left(\frac{k - r^{\frac{2-n}{2}}}{k + r^{\frac{2-n}{2}}} \right), \end{cases}$$

where $k, k_1, k_2 \in \mathbb{R}_+^*$.

Note that the metric given in Corollary 1.5 is shaped as follows

$$\tilde{g} = \frac{1}{k_1^2 \left(k + r^{\frac{2-n}{2}} \right)^{\frac{4}{2-n}}} \sum_{i=1}^n \varepsilon_i dx_i \otimes dx_i - k_2^{(n-2)k} \left(\frac{k - r^{\frac{2-n}{2}}}{k + r^{\frac{2-n}{2}}} \right)^2 dt^2. \quad (1.10)$$

In the Riemannian case when $n = 3$, a particular choice of constants leads us to the Schwarzschild metric. In fact, taking $k^4 k_1^{-2} = k_2 = 1$, we get

$$\tilde{g} = \left(1 + \frac{\mu}{2r^{\frac{1}{2}}} \right)^4 (dx_1^2 + dx_2^2 + dx_3^2) - \left(\frac{1 - \frac{\mu}{2r^{\frac{1}{2}}}}{1 + \frac{\mu}{2r^{\frac{1}{2}}}} \right)^2 dt^2, \quad (1.11)$$

where $\mu = \frac{2}{k}$ represents the gravitating mass and $r = x_1^2 + x_2^2 + x_3^2$. More details in dos Santos and Leandro (2019) or Einstein (1939).

In dos Santos and Leandro (2019), by studying static vacuum Einstein spacetimes, the authors obtained the solutions described in Corollary 1.5 and called them generalized Schwarzschild solutions. They noted that the spatial factor, i.e., the base has zero scalar curvature.

Finally, Theorem 1.4 and Corollary 1.5 lead us to Theorem 1.1.

2 Proof of the Main Results

Proof of Theorem 1.2 The idea of the demonstration is to calculate the expressions of the Ricci tensor in coordinates in the warped product metric and then, we use that $(M, \tilde{g})$ is Einstein manifold.

It is well known (see, for example, O'Neill (1983)) that if $X_1, X_2, \dots, X_n \in \mathcal{L}(\mathbb{R}^n)$ and $Y_1, Y_2, \dots, Y_m \in \mathcal{L}(F)$, where $\mathcal{L}(\mathbb{R}^n)$ and $\mathcal{L}(F)$ are respectively the lift of a vector field on $\mathbb{R}^n$ and F to $\mathbb{R}^n \times F$, then

$$\begin{cases} Ric_{\tilde{g}}(X_i, X_j) = Ric_{\tilde{g}}(X_i, X_j) - \frac{m}{h} Hess_{\tilde{g}}h(X_i, X_j), \forall i, j = 1, \dots, n \\ Ric_{\tilde{g}}(X_i, Y_j) = 0, \forall i = 1, \dots, n, j = 1, \dots, m \\ Ric_{\tilde{g}}(Y_i, Y_j) = Ric_{g_F}(Y_i, Y_j) - \tilde{g}(Y_i, Y_j) \left(\frac{\Delta_{\tilde{g}}h}{h} + (m-1) \frac{\tilde{g}(\nabla h, \nabla h)}{h^2} \right), \forall i, j = 1, \dots, m \end{cases} \quad (2.1)$$

If $\tilde{g} = \frac{1}{\psi^2}g$, then (see Arthur 1987)

$$Ric_{\tilde{g}} = \frac{1}{\psi^2} \left\{ (n-2)\psi Hess_g(\psi) + [\psi \Delta_g \psi - (n-1)|\nabla_g \psi|^2]g \right\}.$$

Since $g(X_i, X_j) = \varepsilon_i \delta_{ij}$, we have

$$\begin{cases} Ric_{\tilde{g}}(X_i, X_j) = \frac{1}{\psi} \left\{ (n-2)Hess_g(\psi)(X_i, X_j) \right\}, \forall i \neq j = 1, \dots, n \\ Ric_{\tilde{g}}(X_i, X_i) = \frac{1}{\psi^2} \left\{ (n-2)\psi Hess_g(\psi)(X_i, X_i) + [\psi \Delta_g \psi - (n-1)|\nabla_g \psi|^2]\varepsilon_i \right\}, \forall i = 1, \dots, n. \end{cases}$$

As $Hess_g(\psi)(X_i, X_j) = \psi_{,x_i x_j}$, $\Delta_g \psi = \sum_{k=1}^n \varepsilon_k \psi_{,x_k x_k}$ and $|\nabla_g \psi|^2 = \sum_{k=1}^n \varepsilon_k \psi_{,x_k}^2$, we infer

$$\begin{cases} Ric_{\tilde{g}}(X_i, X_j) = \frac{(n-2)\psi_{,x_i x_j}}{\psi} \forall i \neq j : 1 \dots n \\ Ric_{\tilde{g}}(X_i, X_i) = \frac{(n-2)\psi_{,x_i x_i} + \varepsilon_i \sum_{k=1}^n \varepsilon_k \psi_{,x_k x_k}}{\psi} - (n-1)\varepsilon_i \sum_{k=1}^n \frac{\varepsilon_k \psi_{,x_k}^2}{\psi^2} \end{cases} \quad (2.2)$$

Recall that

$$Hess_{\tilde{g}}(h)(X_i, X_j) = h_{,x_i x_j} - \sum_k \bar{\Gamma}_{ij}^k h_{,x_k},$$

where $\bar{\Gamma}_{ij}^k$ are the Christoffel symbols of the metric $\bar{g}$. For i, j, k distincts, we have

$$\bar{\Gamma}_{ij}^k = 0 \quad \bar{\Gamma}_{ij}^i = -\frac{\psi_{,x_j}}{\psi} \quad \bar{\Gamma}_{ii}^k = \varepsilon_i \varepsilon_k \frac{\psi_{,x_k}}{\psi} \quad \bar{\Gamma}_{ii}^i = -\frac{\psi_{,x_i}}{\psi}$$

therefore,

$$\begin{cases} Hess_{\bar{g}}(h)(X_i, X_j) = h_{,x_i x_j} + \frac{\psi_{,x_j}}{\psi} h_{,x_i} + \frac{\psi_{,x_i}}{\psi} h_{,x_j}, \quad \forall i \neq j = 1 \dots n \\ Hess_{\bar{g}}(h)(X_i, X_i) = h_{,x_i x_i} + 2 \frac{\psi_{,x_i}}{\psi} h_{,x_i} - \varepsilon_i \sum_{k=1}^n \varepsilon_k \frac{\psi_{,x_k}}{\psi} h_{,x_k} \end{cases} \quad (2.3)$$

Substituting (2.2) and (2.3) in the first equation of (2.1), we obtain

$$Ric_{\bar{g}}(X_i, X_j) = \frac{(n-2)\psi_{,x_i x_j}}{\psi} - \frac{m}{h} \left[h_{,x_i x_j} + \frac{\psi_{,x_j}}{\varphi} h_{,x_i} + \frac{\psi_{,x_i}}{\psi} h_{,x_j} \right], \quad \forall i \neq j \quad (2.4)$$

and

$$\begin{aligned} Ric_{\bar{g}}(X_i, X_i) &= \frac{(n-2)\psi_{,x_i x_i} + \varepsilon_i \sum_{k=1}^n \varepsilon_k \psi_{,x_k x_k}}{\psi} - (n-1) \varepsilon_i \sum_{k=1}^n \frac{\varepsilon_k \psi_{,x_k}^2}{\psi^2} \\ &\quad - \frac{m}{h} \left[h_{,x_i x_i} + 2 \frac{\psi_{,x_i}}{\psi} h_{,x_i} - \varepsilon_i \sum_{k=1}^n \varepsilon_k \frac{\psi_{,x_k}}{\psi} h_{,x_k} \right], \quad \forall i. \end{aligned} \quad (2.5)$$

On the other hand,

$$\begin{cases} Ric_{g_F}(Y_i, Y_j) = \lambda_F g_F(Y_i, Y_j) \\ \tilde{g}(Y_i, Y_j) = h^2 g_F(Y_i, Y_j) \\ \Delta_{\bar{g}} h = \psi^2 \sum_{k=1}^n \varepsilon_k h_{,x_k x_k} - (n-2) \psi \sum_{k=1}^n \varepsilon_k \psi_{,x_k} h_{,x_k} \\ \tilde{g}(\nabla h, \nabla h) = \psi^2 \sum_{k=1}^n \varepsilon_k h_{,x_k}^2 \end{cases} \quad (2.6)$$

Substituting (2.6) in the third equation of the system (2.1), we have

$$Ric_{\bar{g}}(Y_i, Y_j) = \gamma_{ij} g_F(Y_i, Y_j), \quad (2.7)$$

where

$$\gamma_{ij} = \lambda_F - h \psi^2 \sum_{k=1}^n \varepsilon_k h_{,x_k x_k} + (n-2) h \psi \sum_{k=1}^n \varepsilon_k \psi_{,x_k} h_{,x_k} - (m-1) \psi^2 \sum_{k=1}^n \varepsilon_k h_{,x_k}^2.$$

Observing that

$$\begin{cases} \tilde{g}(X_i, X_j) = \bar{g}(X_i, X_j) = \frac{\delta_{ij}}{\psi^2} \\ \tilde{g}(X_i, Y_j) = 0 \\ \tilde{g}(Y_i, Y_j) = h^2 g_F(Y_i, Y_j) \end{cases},$$

and using the expressions (2.4), (2.5) and (2.7) in

$$Ric_{\tilde{g}} = \lambda \tilde{g},$$

we have $(M, \tilde{g})$ is an Einstein manifold if and only if the Eqs. (1.4), (1.5) and (1.6) are satisfied. $\square$

Proof of Theorem 1.3 Let $\bar{g} = \psi^{-2}g$ be a conformal metric to pseudo-Euclidean g . We are assuming that $\psi(r)$ and $h(r)$ are functions of r , where $r = \sum_{k=1}^n \varepsilon_k x_k^2$, $r > 0$. Hence, we have

$$\psi_{,x_i} = 2\varepsilon_i x_i \psi', \quad \psi_{,x_i x_i} = 4x_i^2 \psi'' + 2\varepsilon_i \psi', \quad \psi_{,x_i x_j} = 4\varepsilon_i \varepsilon_j x_i x_j \psi''$$

and

$$h_{,x_i} = 2\varepsilon_i x_i h', \quad h_{,x_i x_i} = 4x_i^2 h'' + 2\varepsilon_i h', \quad h_{,x_i x_j} = 4\varepsilon_i \varepsilon_j x_i x_j h''.$$

Substituting these expressions into (1.4), we get

$$\begin{aligned} & (n-2)h(4\varepsilon_i \varepsilon_j x_i x_j \psi'') - m\psi(4\varepsilon_i \varepsilon_j x_i x_j h'') \\ & - m(2\varepsilon_i x_i \psi')(2\varepsilon_j x_j h') - m(2\varepsilon_j x_j \psi')(2\varepsilon_i x_i h') = 0, \end{aligned}$$

which is equivalent to

$$4\varepsilon_i \varepsilon_j [(n-2)h\psi'' - m\psi h'' - 2m\psi' h'] x_i x_j = 0.$$

But, it exists $i \neq j$ such that $x_i x_j \neq 0$ and consequently,

$$(n-2)h\psi'' - m\psi h'' - 2m\psi' h' = 0. \quad (2.8)$$

Similarly, considering (1.5), we get

$$\begin{aligned} & 4\psi [(n-2)h\psi'' - m\psi h'' - 2m\psi' h'] x_i^2 + 2\psi [(n-2)\psi' h - m\psi h' + n\psi' h] \varepsilon_i \\ & + 4\varepsilon_i \sum_{k=1}^n \varepsilon_k x_k^2 [\psi \psi'' h - (n-1)(\psi')^2 h + m\psi \psi' h'] = \lambda \varepsilon_i h. \end{aligned}$$

Using (2.8) and the fact that $r = \sum_{k=1}^n \varepsilon_k x_k^2$, it follows that

$$\psi \left[2(n-1) \psi' h - m \psi h' \right] + 2r \left[\psi \psi'' h - (n-1) (\psi')^2 h + m \psi \psi' h' \right] = \frac{\lambda}{2} h.$$

Applying the same technique again by considering (1.6), we have

$$\sum_{k=1}^n \varepsilon_k \left[-\psi^2 h \left(4x_k^2 h'' + 2\varepsilon_k h' \right) + (n-2) \psi h \left(2\varepsilon_i x_i \psi' \right) \left(2\varepsilon_i x_i h' \right) - (m-1) \psi^2 \left(2\varepsilon_i x_i h' \right)^2 \right] = \lambda h^2 - \lambda_F,$$

or equivalently,

$$-2n \psi^2 h h' + 4r \left[-\psi^2 h h'' + (n-2) \psi \psi' h h' - (m-1) \psi^2 (h')^2 \right] = \lambda h^2 - \lambda_F.$$

□

Proof of Theorem 1.4 Consider the Einstein warped product

$$(M^{n+m}, \tilde{g}) = \left(\mathbb{R}^n \times F^m, \frac{1}{\psi^2} g \oplus h^2 g_F \right),$$

where F^m is an Einstein manifold. According to Pina et al. (2021), when the scalar curvature of $\left(\mathbb{R}^n, \frac{1}{\psi^2} g \right)$ is $K = 0$, we have already obtained that the conformal factor ψ and its derivatives are given by

$$\psi(r) = k_1 B^{\frac{2}{2-n}}, \quad \psi'(r) = k_1 B^{\frac{n}{2-n}} r^{-\frac{n}{2}}, \quad \psi''(r) = \frac{k_1 n}{2} \left(B^{\frac{2(n-1)}{2-n}} r^{-n} - B^{\frac{n}{2-n}} r^{-\frac{(n+2)}{2}} \right), \quad (2.9)$$

where $B = k + r^{\frac{2-n}{2}}$, $k, k_1 \in \mathbb{R}_+^*$. Replacing these expressions in (1.8) and using the fact that $(M, \tilde{g})$ is Ricci-flat ($\lambda = 0$), we obtain

$$\frac{h'}{h} = \frac{(n-2) \left(r^{-\frac{n}{2}} - B^{-1} r^{1-n} \right)}{m \left(B - 2r^{\frac{2-n}{2}} \right)}, \quad (2.10)$$

or equivalently,

$$\frac{h'}{h} = \frac{(n-2)k}{m} \frac{r^{-\frac{n}{2}}}{k^2 - r^{2-n}}. \quad (2.11)$$

By direct integration, we get

$$h(r) = k_2^{\frac{(n-2)k}{m}} \left(\frac{k - r^{\frac{2-n}{2}}}{k + r^{\frac{2-n}{2}}} \right)^{1/m}, \quad k_2 > 0. \quad (2.12)$$

Now, let us replace the expressions obtained for $\psi(r)$ and $h(r)$ in (1.7) and (1.9). The first and second derivative of function $h(r)$ are given by

$$h'(r) = \frac{(n-2)kk_2^{\frac{(n-2)k}{m}}}{m} \left(\frac{k - r^{\frac{2-n}{2}}}{k + r^{\frac{2-n}{2}}} \right)^{1/m} \frac{r^{-\frac{n}{2}}}{(k + r^{\frac{2-n}{2}})(k - r^{\frac{2-n}{2}})} \quad (2.13)$$

and

$$h''(r) = \frac{(n-2)kk_2^{\frac{(n-2)k}{m}}}{2m} \left(\frac{k - r^{\frac{2-n}{2}}}{k + r^{\frac{2-n}{2}}} \right)^{1/m} \frac{1}{(k + r^{\frac{2-n}{2}})^2(k - r^{\frac{2-n}{2}})^2} \left[2(n-2)kr^{-n} - (n-4)r^{\frac{2-3n}{2}} - nk^2r^{-\frac{(n+2)}{2}} + 2(n-2) \left(\frac{1-m}{m} \right) kr^{-n} \right]. \quad (2.14)$$

Replacing these expressions in Eq. (1.7), we get that

$$(n-2)h\psi'' - m\psi h'' - 2m\psi' h' = 0$$

if and only if

$$\psi'' - \frac{1}{(k - r^{\frac{2-n}{2}})^2(k + r^{\frac{2-n}{2}})^2} \left\{ \left[(n-2)k^2r^{-n} - \frac{(n-4)r^{\frac{2-3n}{2}}}{2} - \frac{nk^3r^{-\frac{(n+2)}{2}}}{2} + (n-2) \left(\frac{1-m}{m} \right) k^2r^{-n} \right] \psi + 2(k^3r^{-\frac{n}{2}} - kr^{\frac{4-3n}{2}}) \psi' \right\} = 0,$$

which is equivalent to

$$2\psi''(k + r^{\frac{2-n}{2}})^2(k - r^{\frac{2-n}{2}})^2 - \psi \left[2(n-2)k^2r^{-n} - (n-4)r^{\frac{2-3n}{2}} - nk^3r^{-\frac{(n+2)}{2}} + 2(n-2) \left(\frac{1-m}{m} \right) k^2r^{-n} \right] - 4 \left(k^3r^{-\frac{n}{2}} - kr^{\frac{4-3n}{2}} \right) \psi' = 0.$$

Replacing the expressions found in (2.9) in the expression above, we get

$$\begin{aligned} & nr^{-n}(k + r^{\frac{2-n}{2}})(k - r^{\frac{2-n}{2}})^2 - nr^{-\frac{(n+2)}{2}}(k + r^{\frac{2-n}{2}})^2(k - r^{\frac{2-n}{2}})^2 - 4r^{-\frac{n}{2}} \left(k^3r^{-\frac{n}{2}} - kr^{\frac{4-3n}{2}} \right) \\ & - (k + r^{\frac{2-n}{2}}) \left[2(n-2)k^2r^{-n} - (n-4)kr^{\frac{2-3n}{2}} - nk^3r^{-\frac{(n+2)}{2}} + 2(n-2) \left(\frac{1-m}{m} \right) k^2r^{-n} \right] \\ & = 0 \end{aligned}$$

if and only if

$$\left[nk^3 - 4k^3 + nk^3 - 2(n-2) \left(\frac{1-m}{m} \right) k^3 - 2nk^3 + 4k^3 \right] r^{-n} \\ + \left[-nk^2 + 2nk^2 + nk^2 - 4k^2 - 2nk^2 + 4k^2 - 2(n-2) \left(\frac{1-m}{m} \right) k^2 \right] r^{\frac{2-3n}{2}} = 0$$

if and only if

$$2(n-2) \left(\frac{1-m}{m} \right) k^2 \left(kr^{-n} + r^{\frac{2-3n}{2}} \right) = 0.$$

Since $k \neq 0$, one concludes that

$$(n-2)h\psi'' - m\psi h'' - 2m\psi' h' = 0,$$

if and only if $m = 1$; i.e., fiber F is 1-dimensional.

Using the expressions obtained for ψ and h in (1.9), with $m = 1$ and $\lambda_F = \lambda = 0$, one sees that it is trivially satisfied. This concludes the proof of Theorem 1.4. $\square$

Proof of Corollary 1.5 Functions $\psi(r)$ and $h(r)$ provided by Corollary 1.5 were obtained from demonstration of Theorem 1.4, making $m = 1$ in expression of h . Substituting these functions and their derivatives in Eqs. (1.7), (1.8) and (1.9), a straightforward computation shows that it is trivially satisfied. $\square$

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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