



Rigidity results on gradient Schouten solitons

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ABSTRACT

In this paper, we consider gradient Schouten solitons, conformal to an n -dimensional pseudo-Euclidean space and invariant under the action of an $(n-1)$ -dimensional translation group. We provide all such solutions and further, we prove that, in the Riemannian case, there does not exist a complete gradient Schouten Soliton that is conformal to Euclidean space and invariant to translation. Furthermore, we consider ρ -Einstein solitons of type $M = (B^n, g^*) \times (F^m, g_F)$, where (B^n, g^*) is conformal to a pseudo-Euclidean space and invariant under the action of the pseudo-orthogonal group, and (F^m, g_F) is an Einstein manifold. We provide all the solutions for the gradient Schouten soliton case, in the Riemannian case, we prove that if $M = (B^n, g^*) \times (F^m, g_F)$ is a complete gradient Schouten soliton, then (B^n, g^*) is isometric to $S^{n-1} \times \mathbb{R}$ and F^m is a compact Einstein manifold.

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1. Introduction and main results

In 1982, R. Hamilton introduced a nonlinear evolution equation for Riemannian metrics, aiming to find canonical metrics on manifolds (see [11]). This evolution equation is known as the Ricci flow and has since been widely and successfully used, most notably in Perelman's solution of the Poincaré conjecture. Since then, problems involving Ricci curvature have aroused the interest of many researchers. Various cases involving the Ricci curvature have been studied in the literature, such as gradient Ricci solitons (see, e.g., [5], [7], [13], and their references), gradient Ricci almost solitons (see, e.g., [17]), Einstein-type manifolds (see, e.g., [8]), gradient Schouten solitons and ρ -Einstein solitons (see, e.g., [9]).

Following Catino and Mazzieri in [9], we say that a Riemannian manifold (M^n, g) , $n \geq 3$ is a gradient ρ -Einstein soliton with $\rho \in \mathbb{R}$, $\rho \neq 0$, if there exists a smooth function $h : M \rightarrow \mathbb{R}$, such that the metric g satisfies the equation

$$\text{Ric}_g + \text{Hess}_g h = \rho K_g g + \lambda g \quad (1.1)$$

for some constant $\lambda \in \mathbb{R}$, where K_g is the scalar curvature of the metric g . A ρ -Einstein soliton is said to be shrinking, steady, or expanding if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. Furthermore, a ρ -Einstein soliton is said to be a gradient Einstein soliton, gradient traceless Ricci soliton, or gradient Schouten soliton if $\rho = \frac{1}{2}$, $\rho = \frac{1}{n}$, or $\rho = \frac{1}{2(n-1)}$, respectively.

In particular, this paper, specifically focuses on the gradient Schouten soliton. These are Riemannian manifolds satisfying

$$A_g = \frac{1}{n-2} (\lambda g - \text{Hess}_g h) \quad (1.2)$$

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for some smooth function h and some constant $\lambda \in \mathbb{R}$. Where

$$A_g = \frac{1}{n-2} \left(Ric_g - \frac{K_g}{2(n-1)} g \right) \quad (1.3)$$

is the Schouten tensor of M and K_g is the scalar curvature of the metric g . The Schouten soliton equation (1.2) relates geometric information about the curvature of the manifold through the Schouten tensor and the geometry of the level sets of the potential function by means of their second fundamental form. This tensor has been considered especially in conformal geometry and its importance can be seen in the following decomposition of the curvature tensor

$$R_g = W_g + A_g \odot g, \quad (1.4)$$

where R_g is the curvature tensor, $\odot$ is the Kulkarni-Nomizu product and W_g is the Weyl tensor of the metric g . For more details, see, e.g., [2] and [16]. Since the Weyl tensor is conformally invariant, to study the deformation of the conformal metric, we only need to understand the Schouten tensor.

ρ -Einstein soliton has been recently investigated in both the Riemannian and pseudo-Riemannian contexts, as can be seen in [3,4,9,18]. Nice results established by Catino and Mazzieri in [9] state that every compact gradient Einstein, Schouten, or traceless Ricci soliton, is trivial. In addition, they proved that every complete gradient steady Schouten soliton is trivial, hence Ricci flat. In [18], the authors considered ρ -Einstein solitons that are conformal to a pseudo-Euclidean space and invariant under the action of the pseudo-orthogonal group. They provided all the solutions for the gradient Schouten soliton case. Motivated by Bryant's classification in [6], it was proved that if a gradient Schouten soliton is both complete, conformal to a Euclidean metric, and rotationally symmetric, then it is shrinking and isometric to $\mathbb{S}^{n-1} \times \mathbb{R}$.

Throughout this paper, we consider two related problems involving gradient Schouten solitons. The first problem involves considering gradient Schouten solitons, which are completely rewritten according to an n -dimensional pseudo-Euclidean space, invariant under the action of an $(n-1)$ -dimensional translation group, so we find all such solutions (see, Theorem 1.4). Moreover, in the Riemannian case, we prove that the metrics obtained in Theorem 1.4 are not complete, leading us to the following result.

Corollary 1.1. *Let $(\mathbb{R}^n, \bar{g})$, $n \geq 2$ a gradient Schouten soliton, conformal to the Euclidean metric and invariant under the action of an $(n-1)$ -dimensional translation group, where $\bar{g}_{ij} = \delta_{ij}/\varphi^2$. Then $(\mathbb{R}^n, \bar{g})$ is not complete.*

The second problem consists in considering gradient Schouten solitons on the product manifold $M = (B^n, g^*) \times (F^m, g_F)$, where (B^n, g^*) is conformal to a pseudo-Euclidean space and invariant under the action of the pseudo-orthogonal group, and (F^m, g_F) is an Einstein manifold, equipped with the metric tensor $\tilde{g} = g_B + g_F$. From the standpoint of manifolds, the next result is an extension of the result in [18].

Corollary 1.2. *Let $((M^n, g^*) \times F^m)$, $n \geq 3$ and $m \geq 1$ be a complete gradient Schouten soliton, where (M^n, g^*) is conformal to a Euclidean metric and rotationally symmetric, where $g^*_{ij} = \delta_{ij}/\psi^2$, and F^m is an Einstein Riemannian manifold. Then (M^n, g^*) is isometric to $\mathbb{S}^{n-1} \times \mathbb{R}$, F^m is a compact manifold with Ricci curvature equal to $\lambda_F = (n-2)$ and $\tilde{\lambda} = -\frac{(m-n+1)(n-2)}{2(n-1)}$.*

Based on this result, we can see that depending on the choices of m and n , $\tilde{\lambda}$ can be negative, zero or positive. In this context, it was possible to build explicit rotationally symmetric examples of complete gradient Schouten solitons, whether they are expanding, steady or shrinking. Additionally, we have obtained a class of examples that are not locally conformally flat.

More specifically, let $(\mathbb{R}^n, g)$ be the standard pseudo-Euclidean space with metric g and coordinates $(x_1, \dots, x_n)$, with $g_{ij} = \delta_{ij}\varepsilon_i$, $1 \leq i, j \leq n$, where δ_{ij} is the Kronecker delta, and $\varepsilon_i = \pm 1$. Let $\xi = \sum_{i=1}^n \alpha_i x_i$, $\alpha_i \in \mathbb{R}$ be a basic invariant for an $(n-1)$ -dimensional translation group. Let us get the necessary and sufficient conditions for the existence of differentiable functions $\varphi(\xi)$ and $f(\xi)$ such that the metric $\bar{g} = \frac{g}{\varphi^2}$ satisfies equation (1.2). Initially, we find a system of differential equations that the functions f and φ must satisfy, so that the metric $\bar{g} = \frac{g}{\varphi^2}$ satisfies (1.1) (see Theorem 3.1). It should be noted these conditions differ depending on whether the direction $\alpha = \sum_{k=1}^n \alpha_k \partial/\partial \alpha_k$ being null (lightlike) or not. In the null case, the gradient Schouten soliton is necessarily steady (see Theorem 3.2). In this scenario, we show that there are infinitely many gradient Schouten solitons conformal to the standard pseudo-Euclidean metric, which are invariant under the action of an $(n-1)$ -dimensional group acting on $\mathbb{R}^n$ (Theorem 1.3). Considering $\rho = \frac{1}{2(n-1)}$, in Lemma 3.1, we find the necessary conditions required for the functions to be solutions of the system of ODEs obtained in Corollary 3.3. Finally, we obtain the necessary and sufficient conditions for $f(\xi)$ and $\varphi(\xi)$ for the existence of one gradient Schouten soliton. In this case, all solutions are given explicitly. Moreover, in the Riemannian case, we prove that there does not exist a complete gradient Schouten soliton that is conformal to Euclidean space and invariant to translation. These results are an extension of the result in [1] for the gradient Schouten solitons, with $\lambda \in \mathbb{R}$.

Furthermore, when considering ρ -Einstein solitons as being product manifolds of type

$$M = (B^n, g^*) \times (F^m, g_F),$$

where (B^n, g^*) is conformal to a pseudo-Euclidean space and invariant under the action of the pseudo-orthogonal group, (F^m, g_F) is an Einstein manifold, we assume the potential function is defined only in B^n . More precisely, let $(\mathbb{R}^n, g)$ be the standard pseudo-Euclidean space with metric g and coordinates $(x_1, \dots, x_n)$, with $g_{ij} = \delta_{ij}\varepsilon_i$, $1 \leq i, j \leq n$, where δ_{ij} is the Kronecker delta, and $\varepsilon_i = \pm 1$. Let $r = \sum_{i=1}^n \varepsilon_i x_i^2$ be a basic invariant for an $(n-1)$ -dimensional pseudo-orthogonal group. Let us find the necessary and sufficient conditions for the existence of differentiable functions $\psi(r)$ and $h(r)$ such that the metric $\tilde{g} = g^* + g_F$, where $g^* = \frac{g}{\psi^2}$, satisfies the equation (1.2). Initially, we started by considering a ρ -Einstein soliton as being a product manifold of type $M = (B^n, g^*) \times (F^m, g_F)$, we then find a system of differential equations, which functions h and ψ must satisfy, so that the metric $\tilde{g} = g^* + g_F$ satisfies (1.1) (see Theorem 3.4). Note that if the solutions are invariant under the action of the pseudo-orthogonal group, the system of partial differential equations given in Theorem 3.4 can be transformed into a system of ordinary differential equations (see Theorem 3.5). Setting $\rho = \frac{1}{2(n-1)}$, in Lemma 3.2, we find the necessary conditions on $\psi(r)$ so that the problem allows for a solution (Corollary 3.6).

Finally, we obtain the necessary and sufficient conditions for $h(r)$ and $\psi(r)$ for the existence of a gradient Schouten soliton. In this case, all solutions are given explicitly (Theorem 1.7). Moreover, in the Riemannian case, we show that if a gradient Schouten soliton is complete, where (M^n, g^*) is conformal to an Euclidean metric and rotationally symmetric, where $g_{ij}^* = \delta_{ij}/\psi^2$, and F^m is an Einstein Riemannian manifold, then (M^n, g^*) is isometric to $S^{n-1} \times \mathbb{R}$, F^m is a compact manifold with Ricci curvature equal to $\lambda_F = (n-2)$ and $\tilde{\lambda} = -\frac{(m-n+1)(n-2)}{2(n-1)}$.

Let us establish our main ground. Settle down the metric

$$\tilde{g} = g^* + l^2 g_F$$

on $(\mathbb{R}^n, g^*) \times_l F^m$ (see [12] and the references within), with $m \geq 1$, it is well known that:

$$\begin{cases} Ric_{\tilde{g}}(X, Y) = Ric_{g^*}(X, Y) - \frac{m}{l} Hess_{g^*} l(X, Y) \\ Ric_{\tilde{g}}(V, H) = Ric_{g_F}(V, H) - \left(\frac{\Delta_{g^*} l}{l} + (m-1) \frac{g^*(\nabla_{g^*} l, \nabla_{g^*} l)}{l^2} \right) \tilde{g}(V, H) \\ Ric_{\tilde{g}}(X, V) = 0 \end{cases} \quad (1.5)$$

where $Hess_{g^*}$, ∇_{g^*} and Δ_{g^*} are, respectively, the Hessian, gradient and the Laplacian operators for g^* . Moreover, X and Y are horizontal vector fields, while H and V are vertical vector fields (see [14]). Part of our results will be considered product manifolds of type $(\mathbb{R}^n, g^*) \times F^m$ as being a particular case of the warped product $(\mathbb{R}^n, g^*) \times_l F^m$ considering $l \equiv 1$. For more details, see [14].

The following theorem shows that there are infinitely many gradient steady Schouten solitons that are conformal to the standard pseudo-Euclidean space and are invariant under the action of an $(n-1)$ -dimensional translation group acting on $\mathbb{R}^n$ when $\varepsilon_{i_0} = 0$.

Theorem 1.3. (see [1]) Let $\varphi(\xi)$ be any nonvanishing differentiable function, where $\xi = \sum_{i=1}^n \alpha_i x_i$ and $\varepsilon_{i_0} = 0$. Then function $f(\xi)$ given by:

$$\begin{cases} f(\xi) = -(n-2) \int \frac{1}{\varphi^2} \left(\int \varphi \varphi'' d\xi \right) d\xi, & \text{for } n \geq 3, \\ f(\xi) = k \int \frac{1}{\varphi^2} d\xi + k_1, & \text{for } n = 2, \end{cases} \quad (1.6)$$

where $k \in \mathbb{R}_+^*$ and $k_1 \in \mathbb{R}$, satisfies (3.4) and $(\mathbb{R}^n, \bar{g})$, where $\bar{g} = g/\varphi^2$ is a gradient ρ -Einstein soliton steady with f as a potential function.

Next, we classify all metrics that are invariant under translation that satisfy equation (1.2) with $\lambda \in \mathbb{R}$.

Theorem 1.4. Let $(\mathbb{R}^n, g)$, $n \geq 2$, be a pseudo-Euclidean space with coordinates $x = (x_1, \dots, x_n)$ and metric component $g_{ij} = \delta_{ij}\varepsilon_i$, $1 \leq i, j \leq n$, where $\varepsilon_i = \pm 1$. Considering smooth functions $\varphi(\xi)$ and $f(\xi)$, where $\xi = \sum_{i=1}^n \alpha_i x_i$, $\alpha_i \in \mathbb{R}$ and $\varepsilon_{i_0} \neq 0$. There exists a metric $\bar{g} = \frac{1}{\varphi^2} g$ such that $(\mathbb{R}^n, \bar{g})$ is a gradient Schouten soliton with f as a potential function if, and only if, functions $\varphi(\xi)$ and $f(\xi)$ satisfy

$$\begin{cases} \varphi(\xi) = e^{(k\xi + k_1)}, \\ f(\xi) = \frac{(2-n)k}{2} \xi + \frac{\varepsilon_{i_0} \lambda}{2k^2} e^{-2(k\xi + k_1)} + k_2, \end{cases} \quad (1.7)$$

where $k \in \mathbb{R}_+^*$ and $k_1, k_2 \in \mathbb{R}$.

Remark 1.5. We believe that taking $\rho \neq \frac{1}{2(n-1)}$, the solutions for such equation will be nontrivial and very different from the solutions found in Theorem 1.4 when considering $\rho = \frac{1}{2(n-1)}$.

Remark 1.6. The solutions obtained in Theorems 1.3 and 1.4 when $n = 2$ are in fact gradient Yamabe solitons, because $\text{Ric}_g = Kg$ and equation (1.1) become $\nabla^2 f = ((\rho - 1)K + \lambda)g$ (for more details see, e.g., [15]).

As a consequence of Lemma 3.2, we are going to provide a classification theorem for gradient Schouten solitons of type $M = (\mathbb{R}^n, g^*) \times F^m$, where $(\mathbb{R}^n, g^*)$ is conformal to a pseudo-Euclidean space and invariant by the action of the pseudo-orthogonal group and F^m is an Einstein manifold.

Theorem 1.7. Let $(\mathbb{R}^n, g)$ be a pseudo-Euclidean space, $n \geq 3$, with coordinates $x = (x_1, \dots, x_n)$ and $g_{ij} = \delta_{ij}\varepsilon_i$. Consider the product manifold $M = (\mathbb{R}^n, g^*) \times F^m$ with metric tensor $\tilde{g} = g^* + g_F$, where $g^* = \frac{1}{\psi^2}g$, F^m is an Einstein pseudo-Riemannian manifold with $m \geq 2$, constant Ricci curvature λ_F and smooth functions $\psi(r)$ and $h(r)$, where $r = \sum_{k=1}^n \varepsilon_k x_k^2$. Then M is a gradient Schouten soliton with h as a potential function if, and only if, functions h and ψ satisfy:

$$\begin{cases} \psi(r) = k_2 r, \\ h(r) = \frac{\lambda_F}{2k_2^2} r^{-1} + k_1, \\ \tilde{\lambda} = -\frac{\lambda_F}{2(n-1)}(m - 2n + 2), \end{cases} \quad (1.8)$$

and

$$\begin{cases} \psi(r) = k_2 r^{\frac{1}{2}}, \\ h(r) = \frac{(n-2)}{8}(\ln r)^2 + c \ln r + c_1, \\ \tilde{\lambda} = -\frac{(m-n+1)(n-2)}{2(n-1)}k_2^2, \\ \lambda_F = (n-2)k_2^2, \end{cases} \quad (1.9)$$

where c, c_1 and $k_1 \in \mathbb{R}$ and $k_2 \in \mathbb{R}_+^*$.

Before we go any further, we provide some examples to illustrate Theorem 1.3 and Theorem 1.7.

2. Examples for the gradient Schouten solitons

Example 2.1. Let $\varphi(\xi) = B \cosh(A\xi)$, where $A, B \in \mathbb{R}$ such that $A \neq 0$ and $B \neq 0$, $\xi = \sum_{i=1}^n \alpha_i x_i$ and $\sum_{i=1}^n \varepsilon_i \alpha_i = 0$. Solving (1.6), we get

$$f(\xi) = \frac{(2-n)}{4AB^2} \tanh(A\xi) \left(AB^2 (\sinh(2A\xi) + 2A\xi) + 4K_2 \right) + \frac{(n-2)}{4} \cosh(2A\xi) + K_1,$$

where K_2 and $K_1 \in \mathbb{R}$. It follows from Theorem 1.3 that $\bar{g} = g/\varphi^2$ is a gradient steady Schouten soliton, with f as a potential function.

Example 2.2. Similarly, if we choose $\varphi(\xi) = \frac{1}{1+\xi^2}$, then it follows from Theorem 1.3 that

$$f(\xi) = \frac{(n-2)(36 \ln(\xi^2 + 1) + (3\xi^5 + 10\xi^3 + 15\xi) \arctan(\xi))}{60} - \frac{(n-2)\xi^2((\xi(3\xi(4A\xi - 1) + 40A) - 9) - 60A\xi)}{60} + B,$$

where $A, B \in \mathbb{R}$, is a potential function of a steady gradient Schouten soliton $\bar{g} = g/\varphi^2$.

Example 2.3. Let $(\mathbb{R}^n, g)$, $n \geq 3$, be the standard Lorentzian space with coordinates $(x_1, \dots, x_n)$ and signature, $\varepsilon_1 = -1$, $\varepsilon_2 = \dots = \varepsilon_n = 1$. Considering $\xi = \sum_{i=1}^n \alpha_i x_i$ such that $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = \alpha_4 = \dots = \alpha_n = 0$ and chose $\varphi = \frac{1}{1+\xi^2}$, let

$$\bar{g} = (1 + \xi)^2 g, \quad \text{and} \quad f = k\xi \left(1 + \frac{2}{3}\xi^2 + \frac{1}{5}\xi^4 \right) + k_1 \quad (2.1)$$

where $k \in \mathbb{R}_+^*$ and $k_1 \in \mathbb{R}$. Then, we saw that $(\mathbb{R}^n, \bar{g}, f)$ is a steady gradient Schouten soliton with potential function f .

In the Theorem 1.7 depending on the choice of m and n , $\tilde{\lambda}$ can be negative, zero or positive. In this case, it is possible to build examples of complete gradient schouten soliton, either expanding, steady or shrinking.

Example 2.4. The product manifold $(\mathbb{S}^2 \times \mathbb{R}) \times (\mathbb{S}^2 \times \mathbb{S}^2)$, with the usual product metric, is a complete expanding gradient Schouten soliton with $\tilde{\lambda} = -\frac{1}{2}$. In fact, considering $m = 4$ and $n = 3$, and since $S^2 \times S^2$ is an Einstein manifold with $\lambda_F = 1$, it meets the conditions of Theorem 1.7.

Example 2.5. The product manifold $(\mathbb{S}^{n-1} \times \mathbb{R}) \times \mathbb{S}^{n-1}$, with the usual product metric, is a complete steady gradient Schouten soliton with $\tilde{\lambda} = 0$. In fact, considering $m = n - 1$ and since the unitary sphere $(n - 1)$ - dimensional S^{n-1} is an Einstein manifold with $\lambda_F = n - 2$, it meets the conditions of Theorem 1.7.

Example 2.6. The product manifold $(\mathbb{S}^3 \times \mathbb{R}) \times \mathbb{S}_R^2$, with the usual product metric, is a complete shrinking gradient Schouten soliton with $\tilde{\lambda} = \frac{1}{3}$. In fact, considering $m = 2$ and $n = 4$, and since the sphere S_R^2 , with $R = \frac{\sqrt{2}}{2}$, is a Einstein manifold with $\lambda_F = 2$, it meets the conditions of Theorem 1.7.

Example 2.7. As a consequence of Corollary 1.2 we can build examples of complete gradient Schouten solitons that are not locally flat. To do this, just consider $(\mathbb{S}^{n-1} \times \mathbb{R}) \times F^m$ where F^m is a non- trivial compact Einstein manifold with Ricci curvature equal to $\lambda_F = (n - 2)$.

3. Background

In this section, we will present some preliminary results that will be useful in establishing the main results. In what follows, we denote the second order derivative of φ , ψ , f and h by $\varphi_{,x_i x_j}$, $\psi_{,x_i x_j}$, $f_{,x_i x_j}$ and $h_{,x_i x_j}$, respectively, with respect to $x_i x_j$.

Theorem 3.1. (see [18]) Let $(\mathbb{R}^n, g)$, $n \geq 2$, be a pseudo-Euclidean space with coordinates $x = (x_1, \dots, x_n)$ and $g_{ij} = \delta_{ij} \varepsilon_i$, $1 \leq i, j \leq n$, where $\varepsilon_i = \pm 1$. Consider a smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. There exists a metric $\bar{g} = \frac{1}{\varphi^2} g$ such that $(\mathbb{R}^n, \bar{g})$ is a gradient ρ -Einstein soliton with f as a potential function if, and only if, functions φ and f satisfy

$$(n-2)\varphi_{,x_i x_j} + \varphi f_{,x_i x_j} + \varphi_{,x_i} f_{,x_j} + \varphi_{,x_j} f_{,x_i} = 0, \quad i \neq j \quad (3.1)$$

and for each i

$$\begin{aligned} & \varphi \left[(n-2)\varphi_{,x_i x_i} + \varphi f_{,x_i x_i} + 2\varphi_{,x_i} f_{,x_i} \right] \\ & + \varepsilon_i \sum_{k=1}^n \varepsilon_k \left[(n-1) \left(\rho n \varphi_{,x_k}^2 - 2\rho \varphi \varphi_{,x_k x_k} - \varphi_{,x_k}^2 \right) - \varphi \varphi_{,x_k} f_{,x_k} + \varphi \varphi_{,x_k x_k} \right] = \lambda \varepsilon_i. \end{aligned} \quad (3.2)$$

We want to find solutions of the system (3.1), (3.2) of the form $\varphi(\xi)$ and $f(\xi)$, where $\xi = \sum_{i=1}^n \alpha_i x_i$, $\alpha_i \in \mathbb{R}$. Whenever $\sum_{i=1}^n \varepsilon_i \alpha_i^2 \neq 0$, without loss of generality, we may consider $\sum_{i=1}^n \varepsilon_i \alpha_i^2 = \varepsilon_{i_0} = \pm 1$. The next theorem reduced the systems of partial differential equations (3.1), (3.2) to an system of ordinary differential equations.

Theorem 3.2. Let $(\mathbb{R}^n, g)$, $n \geq 2$, be a pseudo-Euclidean space with coordinates $x = (x_1, \dots, x_n)$ and metric components $g_{ij} = \delta_{ij} \varepsilon_i$, $1 \leq i, j \leq n$, where $\varepsilon_i = \pm 1$. Consider the smooth functions $\varphi(\xi)$ and $f(\xi)$, where $\xi = \sum_{i=1}^n \alpha_i x_i$, $\alpha_i \in \mathbb{R}$. Then there exists a metric $\bar{g} = g/\varphi^2$ such that $(\mathbb{R}^n, \bar{g})$ is a gradient ρ -Einstein soliton with f as a potential function, if and only if, functions φ and f satisfy

$$\begin{cases} (n-2)\varphi'' + \varphi f'' + 2\varphi' f' = 0, \\ \varepsilon_{i_0} \left\{ (n-1) \left[(\rho n - 1) (\varphi')^2 - 2\rho \varphi \varphi'' \right] - \varphi \varphi' f' + \varphi \varphi'' \right\} = \lambda, \end{cases} \quad (3.3)$$

whenever $\sum_{k=1}^n \varepsilon_k \alpha_k^2 = \varepsilon_{k_0}$, and

$$(n-2)\varphi'' + \varphi f'' + 2\varphi' f' = 0, \quad \text{and} \quad \lambda = 0, \quad (3.4)$$

whenever $\sum_{k=1}^n \varepsilon_k \alpha_k^2 = 0$.

Proof. Let $\bar{g} = g/\varphi^2$ be a conformal metric of g . We assume that $f(\xi)$ and $\varphi(\xi)$ are functions of ξ . Hence, we have

$$\varphi_{,x_i} = \varphi' \alpha_i, \quad \varphi_{,x_i x_i} = \varphi'' \alpha_i^2, \quad \varphi_{,x_i x_j} = \varphi'' \alpha_i \alpha_j$$

and

$$f_{,x_i} = f' \alpha_i, \quad f_{,x_i x_i} = f'' \alpha_i^2, \quad f_{,x_i x_j} = f'' \alpha_i \alpha_j.$$

Then, by equation (3.1) from Theorem 3.1 implies that,

$$(n-2)\varphi''\alpha_i\alpha_j + \varphi f''\alpha_i\alpha_j + \varphi'\alpha_i f'\alpha_j + \varphi'\alpha_j f'\alpha_i = 0, \quad i \neq j$$

is equivalent to

$$[(n-2)\varphi'' + \varphi f'' + 2\varphi' f']\alpha_i\alpha_j = 0, \quad i \neq j.$$

If $i \neq j$ exists such that $\alpha_i\alpha_j \neq 0$, we have

$$(n-2)\varphi'' + \varphi f'' + 2\varphi' f' = 0.$$

Similarly, considering equation (3.2) from Theorem 3.1, we get the following:

$$\begin{aligned} & \varphi [(n-2)\varphi'' + \varphi f'' + 2\varphi' f'] \alpha_i^2 \\ & + \varepsilon_i \sum_{k=1}^n \varepsilon_k \left\{ (n-1) \left[(\rho n - 1) (\varphi')^2 \alpha_k^2 - 2\rho\varphi\varphi'' \alpha_k^2 \right] - \varphi\varphi' \alpha_k f' \alpha_k + \varphi\varphi'' \alpha_k^2 \right\} = \lambda \varepsilon_i. \end{aligned}$$

It should be noted that $(n-2)\varphi'' + \varphi f'' + 2\varphi' f' = 0$. Therefore,

$$\sum_{k=1}^n \varepsilon_k \alpha_k^2 \left\{ (n-1) \left[(\rho n - 1) (\varphi')^2 - 2\rho\varphi\varphi'' \right] - \varphi\varphi' f' + \varphi\varphi'' \right\} = \lambda.$$

If, for all $i \neq j$, we have $\alpha_i\alpha_j = 0$, then $\xi = x_{i_0}$ and equation (3.1) is trivially satisfied for $i \neq j$. Considering (3.2) for $i \neq i_0$, we have

$$\sum_{k=1}^n \varepsilon_k \left((n-1) \left((\rho n - 1) (\varphi_{x_k})^2 - 2\rho\varphi\varphi_{,x_k x_k} \right) - \varphi\varphi_{,x_k} f_{,x_k} + \varphi\varphi_{,x_k x_k} \right) = \lambda.$$

Since $\xi = x_{i_0}$, we obtain

$$\varphi_{,x_{i_0}} = \varphi', \quad \varphi_{,x_{i_0} x_{i_0}} = \varphi'', \quad f_{,x_{i_0}} = f', \quad \text{and} \quad f_{,x_{i_0} x_{i_0}} = f''.$$

Consequently,

$$\varepsilon_{i_0} \left((n-1) \left((\rho n - 1) (\varphi')^2 - 2\rho\varphi\varphi' \right) - \varphi\varphi' f' + \varphi\varphi'' \right) = \lambda.$$

Therefore, the second equation of (3.3) is satisfied. Considering $i = i_0$ in (3.2), we have

$$(n-2)\varphi\varphi'' + \varphi^2 f'' + 2\varphi\varphi' f' + \left[(n-1) \left((\rho n - 1) (\varphi')^2 - 2\rho\varphi\varphi'' \right) - \varphi\varphi' f' + \varphi\varphi'' \right] = \lambda \varepsilon_{i_0}.$$

Combining the first equation of (3.3) with this equation, the following result. $\square$

However, when $\rho = \frac{1}{2(n-1)}$ and $\varepsilon_{i_0} \neq 0$, the equations found in Theorem 3.2 are easier to deal with, under these considerations we get the following result:

Corollary 3.3. Let $(\mathbb{R}^n, g)$, $n \geq 2$, be a pseudo-Euclidean space with coordinates $x = (x_1, \dots, x_n)$ and metric components $g_{ij} = \delta_{ij}\varepsilon_i$, $1 \leq i, j \leq n$, where $\varepsilon_i = \pm 1$. Consider the smooth functions $\varphi(\xi)$ and $f(\xi)$, where $\xi = \sum_{i=1}^n \alpha_i x_i$, $\alpha_i \in \mathbb{R}$. Then there exists a metric $\bar{g} = \frac{1}{\varphi^2} g$ such that $(\mathbb{R}^n, \bar{g})$ is a gradient Schouten soliton with f as a potential function, if and only if, functions φ and f satisfy

$$\begin{cases} (n-2)\varphi'' + \varphi f'' + 2\varphi' f' = 0, \\ \varepsilon_{i_0} \{ (2-n) (\varphi')^2 - 2\varphi\varphi' f' \} = 2\lambda. \end{cases} \quad (3.5)$$

The next Lemma provides a necessary condition for system (3.5) to admit solutions, whenever that $\varepsilon_{i_0} \neq 0$.

Lemma 3.1. Considering that the smooth functions $\varphi(\xi)$ and $f(\xi)$ are non-constant, such that $\xi = \sum_{i=1}^n \alpha_i x_i$ and $\varepsilon_{i_0} \neq 0$. If $\varphi(\xi)$ and $f(\xi)$ satisfy (3.5), then $\varphi(\xi)$ satisfies the ordinary differential equation

$$\varphi''\varphi = (\varphi')^2. \quad (3.6)$$

Proof. If $\sum_{k=1}^n \varepsilon_k \alpha_k^2 = \varepsilon_{i_0}$, by second expression of (3.5) from Corollary 3.3, we obtain

$$\varepsilon_{i_0} \{(2-n)(\varphi')^2 - 2\varphi\varphi'f'\} = 2\lambda.$$

Deriving the above equation, the following that

$$2(2-n)\varphi'\varphi'' - 2\left[(\varphi')^2 f' + \varphi\varphi''f' + \varphi\varphi'f''\right] = 0$$

equivalently,

$$[(2-n)\varphi' - \varphi f']\varphi'' = (\varphi')^2 f' + \varphi\varphi'f''. \quad (3.7)$$

By the first expression of (3.5) from Corollary 3.3, we have

$$\varphi f'' = -[(n-2)\varphi'' + 2\varphi'f']. \quad (3.8)$$

Substituting this equation into (3.7), we get

$$[(2-n)\varphi' - \varphi f']\varphi'' = (\varphi')^2 f' - \varphi'[(n-2)\varphi'' + 2\varphi'f'],$$

implies that

$$\varphi\varphi'' = (\varphi')^2.$$

Therefore, the proof is done. $\square$

Considering a ρ -Einstein soliton as being product manifolds of type $M = (B^n, g^*) \times (F^m, g_F)$, where (B^n, g^*) is conformal to a pseudo-Euclidean space and invariant under the action of the pseudo-orthogonal group, we get:

Theorem 3.4. Let $(\mathbb{R}^n, g)$ be a pseudo-Euclidean space, $n \geq 3$ with coordinates $x = (x_1, \dots, x_n)$ and $g_{ij} = \delta_{ij}\varepsilon_i$. Consider the product manifold $M = (\mathbb{R}^n, g^*) \times F^m$ with metric tensor $\tilde{g} = g^* + g_F$, where $g^* = \frac{1}{\psi^2}g$, F is an Einstein pseudo-Riemannian manifold with constant Ricci curvature λ_F , $m \geq 1$, $\psi, h : \mathbb{R}^n \rightarrow \mathbb{R}$, are smooth functions. Then M is a gradient ρ -Einstein soliton if, and only if, functions ψ and h satisfy

$$(n-2)\psi_{x_i x_j} + \psi h_{,x_i x_j} + \psi_{,x_i} h_{,x_j} + \psi_{,x_j} h_{,x_i} = 0, \quad i \neq j, \quad (3.9)$$

and for each i

$$\begin{aligned} \varepsilon_i \sum_{k=1}^n \varepsilon_k \left[\psi \psi_{,x_k x_k} - (n-1)(\psi_{,x_k})^2 - \psi \psi_{,x_k} h_{,x_k} - 2(n-1)\rho \psi \psi_{,x_k x_k} + (n-1)n\rho (\psi_{,x_k})^2 \right] \\ + \left[(n-2)\psi \psi_{,x_i x_i} + \psi^2 h_{,x_i x_i} + 2\psi \psi_{,x_i} h_{,x_i} \right] = (\lambda_F m \rho + \tilde{\lambda}) \varepsilon_i, \end{aligned} \quad (3.10)$$

and

$$\sum_{k=1}^n \varepsilon_k \left((n-1)n\rho (\psi_{,x_k})^2 - 2(n-1)\rho \psi \psi_{,x_k x_k} \right) = \lambda_F (m\rho - 1) + \tilde{\lambda}. \quad (3.11)$$

Proof. Suppose initially that $M = (\mathbb{R}^n, g^*) \times F^m$ is a gradient ρ -Einstein soliton with potential function h , that is,

$$\text{Ric}_{\tilde{g}} + \text{Hess}_{\tilde{g}} h = \left(\rho K_{\tilde{g}} + \tilde{\lambda} \right) \tilde{g}, \quad \rho \in \mathbb{R}. \quad (3.12)$$

It follows from equations (1.5) that, if $X_1, X_2, \dots, X_n \in \mathcal{L}(\mathbb{R}^n)$ and $Y_1, Y_2, \dots, Y_m \in \mathcal{L}(F)$, where $\mathcal{L}(\mathbb{R}^n)$ and $\mathcal{L}(F)$ are respectively the spaces of lifts of a vector field on $\mathbb{R}^n$ and F to $(\mathbb{R}^n \times F^m)$, then

$$\begin{cases} \text{Ric}_{\tilde{g}}(X_i, X_j) = \frac{(n-2)\psi_{,x_i x_j}}{\psi}, \quad \forall \quad i \neq j, \quad i, j = 1, \dots, n \\ \text{Ric}_{\tilde{g}}(X_i, X_i) = \frac{1}{\psi} \left[(n-2)\psi_{,x_i x_i} + \varepsilon_i \sum_{k=1}^n \varepsilon_k \psi_{,x_k x_k} \right] - \frac{(n-1)\varepsilon_i}{\psi^2} \sum_{k=1}^n (\psi_{,x_k})^2, \quad \forall \quad i = 1, \dots, n \\ \text{Ric}_{\tilde{g}}(X_i, Y_j) = 0, \quad \forall i = 1, \dots, n, j = 1, \dots, m \\ \text{Ric}_{\tilde{g}}(Y_i, Y_j) = \lambda_F \tilde{g}(Y_i, Y_j), \quad \forall \quad i, j = 1, \dots, m. \end{cases} \quad (3.13)$$

As the potential function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined only on the basis, we have

$$\text{Hess}_{\tilde{g}}(h)(X_i, X_j) = \text{Hess}_{g^*}(h)(X_i, X_j) \quad \forall i, j = 1, \dots, n.$$

Therefore,

$$\begin{cases} \text{Hess}_{\tilde{g}}h(X_i, X_j) = h_{,X_i X_j} + \frac{\psi_{,X_i} h_{,X_j}}{\psi} + \frac{\psi_{,X_j} h_{,X_i}}{\psi}, \quad \forall i \neq j, \quad i, j = 1, \dots, n \\ \text{Hess}_{\tilde{g}}(X_i, X_i) = h_{,X_i X_i} + \frac{2\psi_{,X_i} h_{,X_i}}{\psi} - \varepsilon_i \frac{1}{\psi} \sum_{k=1}^n \varepsilon_k \psi_{,X_k} h_{,X_k}, \quad \forall i = 1, \dots, n. \end{cases} \quad (3.14)$$

Now, we split the proof into two cases. In the first case, we will consider the vector fields on the base and in the second one, on the fiber.

If $X_1, X_2, \dots, X_n \in \mathcal{L}(\mathbb{R}^n)$, then to $i \neq j$, from equation (3.12) together with equations (3.13) and (3.14), it follows that

$$\frac{(n-2)\psi_{,X_i X_j}}{\psi} + h_{,X_i X_j} + \frac{\psi_{,X_i} h_{,X_j}}{\psi} + \frac{\psi_{,X_j} h_{,X_i}}{\psi} = 0.$$

Therefore,

$$(n-2)\psi_{X_i X_j} + \psi h_{,X_i X_j} + \psi_{,X_i} h_{,X_j} + \psi_{,X_j} h_{,X_i} = 0.$$

For $i = j$ we will need the following statements. It is well known that (see, e.g., [10])

$$K_{\tilde{g}} = K_{g^*} + K_F \quad (3.15)$$

where $K_{\tilde{g}}$, K_{g^*} and K_F represent the scalar curvatures of $M = (\mathbb{R}^n, g^*) \times F^m$, $(\mathbb{R}^n, g^*)$ and F^m , respectively. Note that

$$K_{g^*} = \sum_{k=1}^n \varepsilon_k \left[2(n-1)\psi \psi_{,X_k X_k} - (n-1)n(\psi_{,X_k})^2 \right], \quad K_F = \lambda_F m.$$

Substituting this expression into (3.15), we get

$$K_{\tilde{g}} = \sum_{k=1}^n \varepsilon_k \left[2(n-1)\psi \psi_{,X_k X_k} - (n-1)n(\psi_{,X_k})^2 \right] + \lambda_F m.$$

Multiplying by ρ on both sides of the above equality, we obtain

$$\rho(K_{g^*} + K_F) = \rho \lambda_F m + \rho \sum_{k=1}^n \varepsilon_k \left[2(n-1)\psi \psi_{,X_k X_k} - (n-1)n(\psi_{,X_k})^2 \right]. \quad (3.16)$$

Replacing the expressions found in (3.13), (3.14) and (3.15) into (3.12), we have

$$\begin{aligned} & \frac{1}{\psi} \left[(n-2)\psi_{,X_i X_i} + \varepsilon_i \sum_{k=1}^n \varepsilon_k \psi_{,X_k X_k} \right] - \frac{(n-1)\varepsilon_i}{\psi^2} \sum_{k=1}^n \varepsilon_k (\psi_{,X_k})^2 + h_{,X_i X_i} + 2\frac{\psi_{,X_i} h_{,X_i}}{\psi} - \frac{\varepsilon_i}{\psi} \sum_{k=1}^n \varepsilon_k \psi_{,X_k} h_{,X_k} \\ &= \left(\rho(K_{g^*} + K_F) + \tilde{\lambda} \right) g^*(X_i, X_i). \end{aligned} \quad (3.17)$$

Substituting equation (3.16) into (3.17) and multiplying by ψ^2 , we can verify that

$$\begin{aligned} & \varepsilon_i \sum_{k=1}^n \varepsilon_k \left[\psi \psi_{,X_k X_k} - (n-1)(\psi_{,X_k})^2 - \psi \psi_{,X_k} h_{,X_k} - 2(n-1)\rho \psi \psi_{,X_k X_k} + (n-1)n\rho(\psi_{,X_k})^2 \right] \\ &+ \left[(n-2)\psi \psi_{,X_i X_i} + \psi^2 h_{,X_i X_i} + 2\psi \psi_{,X_i} h_{,X_i} \right] = (\lambda_F m \rho + \tilde{\lambda}) \varepsilon_i. \end{aligned}$$

If $Y_1, Y_2, \dots, Y_m \in \mathcal{L}(F)$, we get

$$\text{Hess}_{\tilde{g}}(h)(Y_i, Y_j) = 0. \quad (3.18)$$

Substituting the equations given by (3.13), (3.16) and (3.18) into (3.12), we obtain

$$\left(\rho \lambda_F m + \rho \sum_{k=1}^n \varepsilon_k \left(2(n-1)\psi \psi_{,X_k X_k} - (n-1)n(\psi_{,X_k})^2 \right) + \tilde{\lambda} \right) g_F(Y_i, Y_j) = \lambda_F g_F(Y_i, Y_j), \quad (3.19)$$

which implies

$$\rho \lambda_F m + \rho \sum_{k=1}^n \varepsilon_k \left(2(n-1) \psi \psi_{,x_k x_k} - (n-1)n (\psi_{,x_k})^2 \right) + \tilde{\lambda} = \lambda_F.$$

Therefore,

$$\sum_{k=1}^n \varepsilon_k \left((n-1)n \rho (\psi_{,x_k})^2 - 2(n-1) \rho \psi \psi_{,x_k x_k} \right) = \lambda_F (m\rho - 1) + \tilde{\lambda}. \quad \square$$

We want to find solutions to the system of equations (3.9), (3.10) and (3.11) of the form $\psi(r)$ and $h(r)$, where $r = \sum_{k=1}^n \varepsilon_k x_k^2$. The following theorem reduces the system of partial differential equations (3.9), (3.10) and (3.11) into a system of ordinary differential equations that must be satisfied by such solutions.

Theorem 3.5. Let $(\mathbb{R}^n, g)$ be a pseudo-Euclidean space, $n \geq 3$, with coordinates $x = (x_1, \dots, x_n)$ and $g_{ij} = \delta_{ij} \varepsilon_i$. Consider $M = (\mathbb{R}^n, g^*) \times F^m$, where $g^* = \frac{1}{\psi^2} g$, F^m an Einstein pseudo-Riemannian manifold with constant Ricci curvature λ_F and smooth functions $\psi(r)$ and $h(r)$, where $r = \sum_{k=1}^n \varepsilon_k x_k^2$. Then M is a gradient ρ -Einstein soliton with h as a potential function if, and only if, functions h and ψ satisfy:

$$(n-2)\psi'' + \psi h'' + 2\psi' h' = 0, \quad (3.20)$$

$$2\psi [2(n-1)(1-n\rho)\psi' + \psi h'] + \quad (3.21)$$

$$4r \left[(1-2(n-1)\rho) \psi \psi'' + (n-1)(n\rho-1) (\psi')^2 - \psi \psi' h' \right] = (\lambda_F m\rho + \tilde{\lambda}).$$

and

$$-4n(n-1)\rho \psi \psi' + 4r \left[(n-1)n\rho (\psi')^2 - 2(n-1)\rho \psi \psi'' \right] = \lambda_F (m\rho - 1) + \tilde{\lambda}. \quad (3.22)$$

Proof. Let $g^* = \psi^{-2} g$ be a conformal metric of g . We are assuming that $\psi(r)$ and $h(r)$ are functions of r , where $r = \sum_{k=1}^n \varepsilon_k x_k^2$. Hence, we have

$$\psi_{,x_i} = 2\varepsilon_i x_i \psi', \quad \psi_{,x_i x_i} = 4x_i^2 \psi'' + 2\varepsilon_i \psi', \quad \psi_{,x_i x_j} = 4\varepsilon_i \varepsilon_j x_i x_j \psi'',$$

and

$$h_{,x_i} = 2\varepsilon_i x_i h', \quad h_{,x_i x_i} = 4x_i^2 h'' + 2\varepsilon_i h', \quad h_{,x_i x_j} = 4\varepsilon_i \varepsilon_j x_i x_j h''.$$

Substituting these expressions into the first equation of Theorem 3.4, we get

$$(n-2) (4\varepsilon_i \varepsilon_j x_i x_j \psi'') + \psi (4\varepsilon_i \varepsilon_j x_i x_j h'') + (2\varepsilon_i x_i \psi') (2\varepsilon_j x_j h') + (2\varepsilon_j x_j \psi') (2\varepsilon_i x_i h') = 0,$$

equivalently,

$$4\varepsilon_i \varepsilon_j [(n-2)\psi'' + \psi h'' + 2\psi' h'] x_i x_j = 0.$$

Since there exists $i \neq j$, such that $x_i x_j \neq 0$, it follows that

$$(n-2)\psi'' + \psi h'' + 2\psi' h' = 0$$

Similarly, considering the second equation of Theorem 3.4, we obtain

$$\begin{aligned} & \psi \left[(n-2) (4x_i^2 \psi'' + 2\varepsilon_i \psi') + \psi (4x_i^2 h'' + 2\varepsilon_i h') + 2(2\varepsilon_i x_i \psi') (2\varepsilon_i x_i h') \right] + \\ & \varepsilon_i \sum_{k=1}^n \varepsilon_k \left[(1-2(n-1)\rho) \psi (4x_k^2 \psi'' + 2\varepsilon_k \psi') + (n-1)(n\rho-1) (2\varepsilon_k x_k \psi')^2 \right] + \\ & \varepsilon_i \sum_{k=1}^n \varepsilon_k \left[-\psi (2\varepsilon_k x_k \psi') (2\varepsilon_k x_k h') \right] = (\lambda_F m\rho + \tilde{\lambda}) \varepsilon_i, \end{aligned}$$

equivalently,

$$\begin{aligned}
& 4\varepsilon_i \sum_{k=1}^n \varepsilon_k x_k^2 \left[(1 - 2(n-1)\rho) \psi \psi'' + (n-1)(n\rho-1) (\psi')^2 - \psi \psi' h' \right] + \\
& 4\psi \left[(n-2)\psi'' + \psi h'' + 2\psi' h' \right] x_i^2 + 2\psi \left[(n-2)\psi' + \psi h' \right] \varepsilon_i + \\
& 2\varepsilon_i \sum_{k=1}^n \varepsilon_k \left[(1 - 2(n-1)\rho) \psi \psi' \varepsilon_k \right] = (\lambda_F m \rho + \tilde{\lambda}) \varepsilon_i.
\end{aligned}$$

From equation (3.20), it follows that

$$\begin{aligned}
& 2\psi \left[(n-2)\psi' + \psi h' + (1 - 2(n-1)\rho) n\psi' \right] + \\
& + 4r \left[(1 - 2(n-1)\rho) \psi \psi'' + (n-1)(n\rho-1) (\psi')^2 - \psi \psi' h' \right] = (\lambda_F m \rho + \tilde{\lambda}),
\end{aligned}$$

implies that

$$\begin{aligned}
& 2\psi \left[2(n-1)(1 - n\rho)\psi' + \psi h' \right] + \\
& 4r \left[(1 - 2(n-1)\rho) \psi \psi'' + (n-1)(n\rho-1) (\psi')^2 - \psi \psi' h' \right] = (\lambda_F m \rho + \tilde{\lambda}).
\end{aligned}$$

Finally, from the last equation of Theorem 3.4, we have

$$\sum_{k=1}^n \varepsilon_k \left[-2(n-1)\rho \psi \left(4x_k^2 \psi'' + 2\varepsilon_k \psi' \right) + (n-1)n\rho (2\varepsilon_k x_k \psi')^2 \right] = \lambda_F (m\rho - 1) + \tilde{\lambda}.$$

Therefore,

$$-4n(n-1)\rho \psi \psi' + 4r \left[(n-1)n\rho (\psi')^2 - 2(n-1)\rho \psi \psi'' \right] = \lambda_F (m\rho - 1) + \tilde{\lambda}. \quad \square$$

Considering $\rho = \frac{1}{2(n-1)}$, the equations found in Theorem 3.5 are easier to deal with, under these considerations we obtain the following result:

Corollary 3.6. Let $(\mathbb{R}^n, g)$ be a pseudo-Euclidean space, $n \geq 3$, with coordinates $x = (x_1, \dots, x_n)$ and $g_{ij} = \delta_{ij}\varepsilon_i$. Consider $M = (\mathbb{R}^n, g^*) \times F^m$, where $g^* = \frac{1}{\psi^2}g$, F^m is an Einstein pseudo-Riemannian manifold with constant Ricci curvature λ_F and smooth functions $\psi(r)$ and $h(r)$, where $r = \sum_{k=1}^n \varepsilon_k x_k^2$. Then M is a gradient Schouten soliton with h as a potential function if, and only if, functions h and ψ satisfy:

$$(n-2)\psi'' + \psi h'' + 2\psi' h' = 0, \quad (3.23)$$

$$\psi \left[(n-2)\psi' + \psi h' \right] + r \left[(2-n)(\psi')^2 - 2\psi \psi' h' \right] = \frac{\lambda_F m}{4(n-1)} + \frac{\tilde{\lambda}}{2}, \quad (3.24)$$

and

$$-n\psi \psi' + 2r \left[\frac{n}{2} (\psi')^2 - \psi \psi'' \right] = \frac{\lambda_F}{4(n-1)} (m - 2n + 2) + \frac{\tilde{\lambda}}{2}. \quad (3.25)$$

The next lemma provides us with the necessary conditions for the existence of gradient Schouten solitons.

Lemma 3.2. Consider smooth functions $\psi(r)$ and $h(r)$ with $r = \sum_{i=1}^n \varepsilon_i x_i^2$. If $\psi(r)$ and $h(r)$ satisfy (3.23) and (3.24), then $\psi(r)$ satisfies the following ordinary differential equation:

$$r\psi \psi'' = r(\psi')^2 - \psi \psi'. \quad (3.26)$$

Proof. Just follow the same steps as Lemma 3.1, considering the second equation of Corollary 3.6. $\square$

4. Proof of the main results

Proof of Theorem 1.4. By expressing (3.6) of Lemma 3.1, we have

$$\frac{\varphi''}{\varphi'} = \frac{\varphi'}{\varphi} \quad (4.1)$$

implies that

$$\varphi(\xi) = e^{k\xi+k_1}. \quad (4.2)$$

Substituting this equation into the second equation of (3.5), we get

$$f'(\xi) = \frac{(2-n)k}{2} - \frac{\varepsilon_{i_0}\lambda}{k} e^{-2(\xi k+k_1)}. \quad (4.3)$$

Therefore,

$$f(\xi) = \frac{(2-n)k}{2}\xi + \frac{\varepsilon_{i_0}\lambda}{2k^2} e^{-2(\xi k+k_1)} + k_2 \quad (4.4)$$

where $k \in \mathbb{R}_+^*$ and $k_1, k_2 \in \mathbb{R}$.

Reciprocally, considering φ and f given by (4.2) and (4.4) respectively, the first equation of (3.5) is satisfied, i.e.,

$$(n-2)\varphi'' + \varphi f'' + 2\varphi' f' = (n-2)k^2 e^{k\xi+k_1} + 2\lambda\varepsilon_{i_0} e^{-(k\xi+k_1)} + (2-n)k^2 e^{k\xi+k_1} - 2\lambda\varepsilon_{i_0} e^{-(k\xi+k_1)}.$$

Therefore,

$$(n-2)\varphi'' + \varphi f'' + 2\varphi' f' = 0.$$

Furthermore, the second equation of (3.5) is trivially satisfied. This concludes the proof of Theorem 1.4. $\square$

Proof of Corollary 1.1. Follows by Theorem 1.4 that the conformal factor φ is given by $\varphi(\xi) = e^{k\xi+k_1}$, where $\xi = \sum_{k=1}^n \alpha_i x_i$ and $\sum_{k=1}^n \alpha_k^2 = 1$. Without loss of generality, suppose that $\alpha_1 \neq 0$. Let us split the proof into two cases, $\alpha_1 > 0$ and $\alpha_1 < 0$.

Case 1: If $\alpha_1 > 0$, consider the divergent curve $\alpha(t) = (t, 0, \dots, 0)$, $t > 0$ such that $\alpha'(t) = (1, 0, \dots, 0)$, $\varphi(\alpha(t)) = e^{k\alpha_1 t+k_1}$ and $\bar{g}(\alpha'(t), \alpha'(t)) = \frac{1}{e^{2(k\alpha_1 t+k_1)}}$. Consequently,

$$l(\alpha) = \lim_{t \rightarrow \infty} \int e^{-2(k\alpha_1 t+k_1)} dt.$$

Therefore

$$l(\alpha) = -\frac{1}{2k\alpha_1} \lim_{t \rightarrow \infty} e^{-2(k\alpha_1 t+k_1)} + c = c,$$

where c is constant and finite. Therefore $(\mathbb{R}^n, \bar{g})$ is not complete.

Case 2: If $\alpha_1 < 0$, consider the divergent curve $\gamma(t) = -\alpha(t)$, $t > 0$ such that $\gamma'(t) = (-1, 0, \dots, 0)$, $\varphi(\gamma(t)) = e^{-k\alpha_1 t+k_1}$ and $\bar{g}(\gamma'(t), \gamma'(t)) = \frac{1}{e^{2(-k\alpha_1 t+k_1)}}$. Consequently,

$$l(\gamma) = \lim_{t \rightarrow \infty} \int e^{-2(-k\alpha_1 t+k_1)} dt.$$

Therefore,

$$l(\gamma) = \frac{1}{2k\alpha_1} \lim_{t \rightarrow \infty} e^{2(k\alpha_1 t-k_1)} + c_1 = c_1,$$

where c_1 is a constant and finite. Therefore, in both cases $(\mathbb{R}^n, \bar{g})$ is not complete. This concludes the proof of Corollary 1.1. $\square$

Proof of Theorem 1.7. From Lemma 3.2 we know that $\psi(r) = k_2 r^s$ with $k_2 > 0$. Since $\psi(r)$ must satisfy equation (3.25), it follows that

$$-nk_2 r^s k_2 s r^{s-1} + nr k_2^2 s^2 r^{2s-2} - 2rk_2 r^s k_2 s(s-1)r^{s-2} = \frac{\lambda_F}{4(n-1)} (m-2n+2) + \frac{\tilde{\lambda}}{2},$$

equivalently,

$$-nk_2^2sr^{2s-1} + nk_2^2s^2r^{2s-1} - 2k_2^2s(s-1)r^{2s-1} = \frac{\lambda_F}{4(n-1)}(m-2n+2) + \frac{\tilde{\lambda}}{2},$$

which implies

$$s(s-1)(n-2)k_2^2r^{2s-1} = \frac{\lambda_F}{4(n-1)}(m-2n+2) + \frac{\tilde{\lambda}}{2}. \quad (4.5)$$

Since the right side must be constant, we have either $s = 1$ or $s = \frac{1}{2}$.

If $s = 1$, we have $\tilde{\lambda} = -\frac{\lambda_F}{2(n-1)}(m-2n+2)$, by the fact that $\psi(r)$ and h satisfy equation (3.24). Note that

$$\psi(r) = k_2r, \quad \psi'(r) = k_2, \quad \psi''(r) = 0.$$

Consequently

$$(n-2)k_2^2r + k_2^2r^2h' + (2-n)k_2^2r - 2k_2^2r^2h' = \frac{\lambda_F m}{4(n-1)} + \frac{\tilde{\lambda}}{2},$$

implies that

$$h(r) = \left(\frac{\lambda_F m}{4(n-1)k_2^2} + \frac{\tilde{\lambda}}{2k_2^2} \right) r^{-1} + k_1.$$

Thus,

$$h(r) = \frac{\lambda_F}{2k_2^2} r^{-1} + k_1.$$

Reciprocally, the function $\psi(r)$ and $h(r)$ given in (1.8) satisfy (3.23), (3.24) and (3.25). Since

$$h' = -\left(\frac{\lambda_F m}{4(n-1)k_2^2} + \frac{\tilde{\lambda}}{2k_2^2} \right) r^{-2}, \quad h'' = 2\left(\frac{\lambda_F m}{4(n-1)k_2^2} + \frac{\tilde{\lambda}}{2k_2^2} \right) r^{-3}.$$

In this case

$$(n-2)\psi'' + \psi h'' + 2\psi' h' = 2\left(\frac{\lambda_F m}{4(n-1)k_2^2} + \frac{\tilde{\lambda}}{2k_2^2} \right) rr^{-3} + 2\left(-\left(\frac{\lambda_F m}{4(n-1)k_2^2} + \frac{\tilde{\lambda}}{2k_2^2} \right) r^{-2} \right),$$

equivalently,

$$(n-2)\psi'' + \psi h'' + 2\psi' h' = 2\left(\frac{\lambda_F m}{4(n-1)k_2^2} + \frac{\tilde{\lambda}}{2k_2^2} \right) r^{-2} - 2\left(\frac{\lambda_F m}{4(n-1)k_2^2} + \frac{\tilde{\lambda}}{2k_2^2} \right) r^{-2}.$$

Therefore $(n-2)\psi'' + \psi h'' + 2\psi' h' = 0$.

Clearly, in this case the second and third equation hold.

If $s = \frac{1}{2}$, we have by (3.24) and (3.25) that $\frac{\lambda_F}{4(n-1)}(m-2n+2) + \frac{\tilde{\lambda}}{2} = -\frac{(n-2)k_2^2}{4}$ and $\frac{\lambda_F m}{4(n-1)} + \frac{\tilde{\lambda}}{2} = \frac{(n-2)k_2^2}{4}$. Consequently we obtain $\tilde{\lambda} = -\frac{(m-n+1)(n-2)}{2(n-1)}k_2^2$ and $\lambda_F = (n-2)k_2^2$. Substituting $\psi(r) = k_2r^{\frac{1}{2}}$ into (3.23), we obtain $h(r) = \frac{(n-2)}{8}(\ln r)^2 + c \ln r + c_1$, where $c, c_1 \in \mathbb{R}$.

This concludes the proof of Theorem 1.7. $\square$

Proof of Corollary 1.2. Except for homothety we can consider $k_2 = 1$ in Theorem 1.7. It has been proven in [18] that $(\mathbb{R}^n, g^*)$, $g_{ij}^* = \delta_{ij}/\psi^2$, with $\psi(r) = r^s$, is complete only for $s = \frac{1}{2}$. Moreover, $(\mathbb{R}^n \setminus \{0\}, g^* = \frac{1}{r}g_0)$ is isometric to $\mathbb{R} \times \mathbb{S}^{n-1}$. Since $\lambda_F = (n-2) > 0$, it follows from the Bonnet–Myers Theorem that F is compact. $\square$

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Declaration of competing interest

The author does not have conflicts of interest of any kind.

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Data availability

No data was used for the research described in the article.

References

- [1] Ezequiel Barbosa, Romildo Pina, Ketten Tenenblat, On gradient Ricci solitons conformal to a pseudo-Euclidean space, *Isr. J. Math.* 200 (1) (2014) 213–224. MR 3219577.
- [2] Arthur L. Besse, *Einstein Manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition. MR 2371700.
- [3] Valter Borges, Rigidity of Bach-flat gradient schouten solitons, 2021.
- [4] Valter Borges, On complete gradient schouten solitons, *Nonlinear Anal.* 221 (2022) 112883.
- [5] Simon Brendle, Rotational symmetry of self-similar solutions to the Ricci flow, *Invent. Math.* 194 (3) (2013) 731–764. MR 3127066.
- [6] R. Bryant, Local existence of gradient Ricci solitons, unpublished.
- [7] Huai-Dong Cao, Qiang Chen, On locally conformally flat gradient steady Ricci solitons, *Trans. Am. Math. Soc.* 364 (5) (2012) 2377–2391. MR 2888210.
- [8] Giovanni Catino, Paolo Mastrolia, Dario D. Monticelli, Marco Rigoli, On the geometry of gradient Einstein-type manifolds, *Pac. J. Math.* 286 (1) (2017) 39–67. MR 3582400.
- [9] Giovanni Catino, Lorenzo Mazzeri, Gradient Einstein solitons, *Nonlinear Anal.* 132 (2016) 66–94. MR 3433954.
- [10] Paul E. Ehrlich, Yoon-Tae Jung, Seon-Bu Kim, Constant scalar curvatures on warped product manifolds, *Tsukuba J. Math.* 20 (1) (1996) 239–256. MR 1406047.
- [11] Richard S. Hamilton, Three-manifolds with positive Ricci curvature, *J. Differ. Geom.* 17 (1982) 255–306.
- [12] Benedito Leandro, João Paulo dos Santos, Reduction of gradient Ricci soliton equation, *Ann. Acad. Sci. Fenn., Math.* 45 (2) (2020) 1003–1011. MR 4112273.
- [13] Lei Ni, Nolan Wallach, On a classification of gradient shrinking solitons, *Math. Res. Lett.* 15 (5) (2008) 941–955. MR 2443993.
- [14] Barrett O'Neill, *Semi-Riemannian Geometry, Pure and Applied Mathematics*, vol. 103, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983. With applications to relativity. MR 719023.
- [15] Daskalopoulos Panagioti, Natasa Sesum, The classification of locally conformally flat Yamabe solitons, 2012.
- [16] Mauricio Pieterzack, Romildo Pina, Prescribed diagonal schouten tensor in locally conformally flat manifolds, *J. Geom.* 104 (2013) 341–355.
- [17] Stefano Pigola, Marco Rigoli, Michele Rimoldi, Alberto G. Setti, Ricci almost solitons, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 10 (4) (2011) 757–799. MR 2932893.
- [18] Romildo Pina, Ilton Menezes, On gradient Schouten solitons conformal to a pseudo-Euclidean space, *Manuscr. Math.* 163 (3–4) (2020) 395–406. MR 4159803.